

Recursive Formula for the Moments of Queue Length in the $M/M/1$ Queue

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Abstract—This letter presents the recursive formulas of the moments of queue length for the $M/M/1$ queue and $M/M/1/B$ queue, respectively. The higher moments of queue length are important for optimization problem. Our method provides an alternative approach to derive the moments of queue length, instead of taking the derivatives of the moment generating function.

Index Terms— $M/M/1$ queue, queue length distribution, moments, recursive formula.

I. INTRODUCTION

THE $M/M/1$ queue is of great interest in queueing theory because of its concise properties. It has been widely used in the traffic engineering of communication networks [1]-[5]. Much effort has thus been devoted to the study of both the equilibrium and transient properties of this queue [6]-[8]. The higher moments of queue length are important in optimization problems [9]-[11] (e.g. using higher moments to construct some performance bound or to analyze the system transient behavior). Usually, we compute the moments of queue length distribution from its Moment Generating Function (MGF). In this letter, starting from the basic state transition equations of the queue, we get the recursive formula of the moments of queue length for both the $M/M/1$ and the finite $M/M/1/B$ queues. Our method to derive the recursive formulas can also be applied to various of queueing systems to compute the moments of queue length. For example, the $M/M/c$, the queue with c servers; $M/E_r/1$ queue, E_r means the r -stage Erlangian server; or the $MMPP/M/1$ queue, the queue with Markov-Modulated Poisson arrival Process. We will demonstrate the moment recursive formulas of the queue length for the $M/M/1$ and $M/M/1/B$ queues in Section II and Section III, respectively.

II. MOMENT RECURSIVE FORMULA IN $M/M/1$ QUEUE

A. Brief review of $M/M/1$ queue

$M/M/1$ is the Kendall's notation of a single server queue with a Poisson arrival process and an exponential distribution for the service time [12]. Here, we assume the packet

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arrival to the queue is Poisson process with rate λ , and the packet length is exponential distributed with mean $1/\mu$. Define $\{P_n, n = 0, 1, 2, \dots\}$ as the equilibrium probabilities that the queue contains n packets (including the one in service). Then, they satisfy the following state transition equations [12]:

$$\begin{cases} \mu P_1 = \lambda P_0 \\ \mu P_{n+1} = (\lambda + \mu)P_n - \lambda P_{n-1} \end{cases} \quad (n = 1, 2, 3\dots). \quad (1)$$

The offered load to the queue is $\rho = \lambda/\mu$; only if $\rho < 1$, the queue can be stable and approach to the equilibrium state. P_0 is the probability of system being empty, and from Little's law [12], we have

$$1 - P_0 = \frac{\lambda}{\mu} = \rho. \quad (2)$$

Based on equation (1) and (2), we can get the following results for the number of packet in the queue [12]: *Packet number distribution*

$$P_n = \rho(1 - \rho)^n \quad (n = 0, 1, 2, \dots, \infty). \quad (3)$$

Average number of packet

$$\bar{N} = \sum_{n=1}^{\infty} n P_n = \frac{\rho}{1 - \rho}. \quad (4)$$

Moment generating function

$$P(z) = \frac{1 - \rho}{1 - \rho z}. \quad (5)$$

By taking the derivatives of equation (5), we can obtain the moments of queue length. In the following section, we will present the recursive formula to compute the moments of queue length.

B. Recursive formula in $M/M/1$ queue

Theorem 1: In an $M/M/1$ queue with arrival rate λ and mean service time $1/\mu$, when the offered load $\rho = \frac{\lambda}{\mu} < 1$, and define $M_0 = \sum_{n=0}^{\infty} P_n = 1$, then the k th moment $M_k (k \geq 1)$ of queue length satisfies

$$M_k = \frac{\sum_{l=2}^{k+1} \binom{k+l}{l} [(-1)^l + \rho] M_{k+1-l} - (-1)^{k+1} [1 - \rho]}{(k+1)(1 - \rho)}. \quad (6)$$

Actually, from Theorem 1, we get

$$M_1 = \frac{\rho}{1 - \rho}, \quad (7)$$

$$M_2 = \frac{1 + \rho}{1 - \rho} M_1, \quad (8)$$

$$M_3 = \frac{3(1+\rho)}{2(1-\rho)}M_2 - M_1 + \frac{\rho}{2(1-\rho)}, \quad (9)$$

and the arbitrary higher moments can be recursively computed.

Proof:

When $k \geq 1$, define $M_k = \sum_{n=0}^{\infty} n^k P_n$. Multiplying n^{k+1} on both sides of the second equation of (1) yields

$$\mu n^{k+1} P_{n+1} = (\lambda + \mu)n^{k+1} P_n - \lambda n^{k+1} P_{n-1}. \quad (10)$$

We can further write equation (10) as

$$\begin{aligned} & \mu [(n+1) - 1]^{k+1} P_{n+1} \\ & = (\lambda + \mu)n^{k+1} P_n - \lambda [(n-1) + 1]^{k+1} P_{n-1}, \end{aligned} \quad (11)$$

$$\begin{aligned} & \mu \left[\sum_{l=0}^{k+1} \binom{k+1}{l} (n+1)^{k+1-l} (-1)^l \right] P_{n+1} \\ & = (\lambda + \mu)n^{k+1} P_n - \lambda \left[\sum_{l=0}^{k+1} \binom{k+1}{l} (n-1)^{k+1-l} \right] P_{n-1}. \end{aligned} \quad (12)$$

Now let $n = 1, 2, 3, \dots, \infty$, respectively in equation (12) and sum up all these equations together, we obtain

$$\begin{aligned} & \mu \sum_{l=0}^{k+1} \binom{k+1}{l} (-1)^l \left[\sum_{n=1}^{\infty} (n+1)^{k+1-l} P_{n+1} \right] \\ & = (\lambda + \mu) \sum_{n=1}^{\infty} n^{k+1} P_n \\ & \quad - \lambda \sum_{l=0}^{k+1} \binom{k+1}{l} \left[\sum_{n=1}^{\infty} (n-1)^{k+1-l} P_{n-1} \right], \end{aligned} \quad (13)$$

$$\begin{aligned} & \mu \sum_{l=0}^{k+1} \binom{k+1}{l} (-1)^l [M_{k+1-l} - P_1] - (-1)^{k+1} \mu P_0 \\ & = (\lambda + \mu) M_{k+1} - \lambda \sum_{l=0}^{k+1} \binom{k+1}{l} M_{k+1-l}. \end{aligned} \quad (14)$$

Noting that in the left side of (13), n starts from 1 to ∞ , which results in the minus terms related to P_0 and P_1 in (14). Combining (14) with the first equation in (1), we can obtain the formula to calculate M_k :

$$\begin{aligned} M_k & = \frac{\sum_{l=2}^{k+1} \binom{k+1}{l} [(-1)^l \mu + \lambda] M_{k+1-l} - (-1)^{k+1} [\mu - \lambda]}{(k+1)(\mu - \lambda)} \\ & = \frac{\sum_{l=2}^{k+1} \binom{k+1}{l} [(-1)^l + \rho] M_{k+1-l} - (-1)^{k+1} [1 - \rho]}{(k+1)(1 - \rho)}. \end{aligned} \quad (15)$$

Q.E.D.

III. MOMENT RECURSIVE FORMULA IN $M/M/1/B$ QUEUE

A. Brief review of $M/M/1/B$ queue

The $M/M/1/B$ queue can only accommodate at most B packets, including the one in service, if any. The arriving packet finding the system containing B packets will

be blocked. When $B \rightarrow \infty$, we get the $M/M/1$ queue as described in Section II. We can get the corresponding equilibrium state transition equation as [12]:

$$\begin{cases} \mu P_1 = \lambda P_0 \\ \mu P_{n+1} = (\lambda + \mu) P_n - \lambda P_{n-1} \quad (n = 1, 2, 3, \dots, B-1) \\ \mu P_B = \lambda P_{B-1}, \end{cases} \quad (16)$$

where P_0 is the probability of system being empty, P_B is the packet blocking probability of this finite queue. From Little's law, we get

$$1 - P_0 = (1 - P_B) \frac{\lambda}{\mu}. \quad (17)$$

When $\rho < 1$, from equation (16) and (17), we can obtain the following results for the the number of packet [12]:

Packet number distribution

$$P_n = \frac{(1-\rho)\rho^n}{1-\rho^{B+1}} \quad (n = 0, 1, 2, \dots, B). \quad (18)$$

Probability of system being empty

$$P_0 = \frac{1-\rho}{1-\rho^{B+1}}. \quad (19)$$

Packet blocking probability

$$P_B = \frac{(1-\rho)\rho^B}{1-\rho^{B+1}}. \quad (20)$$

Average number of packet

$$\bar{N} = \frac{\rho}{1-\rho} - \frac{(B+1)\rho^{B+1}}{1-\rho^{B+1}}. \quad (21)$$

Moment generating function

$$P(z) = \frac{1-\rho}{1-\rho^{B+1}} \left[\frac{1 - (\rho z)^{B+1}}{1 - \rho z} \right]. \quad (22)$$

By taking the derivatives of equation (22), we can obtain the moments of queue length.

B. Recursive formula in $M/M/1/B$ queue

Theorem 2: In an $M/M/1/B$ queue with arrival rate λ and mean service time $1/\mu$, when the offered load $\rho = \frac{\lambda}{\mu} < 1$, and define $M_0^* = \sum_{n=0}^B P_n = 1$, then the k th moment $M_k^* (k \geq 1)$ of queue length satisfies

$$\begin{aligned} M_k^* & = \frac{\sum_{l=2}^{k+1} \binom{k+l}{l} [(-1)^l + \rho] M_{k+1-l}^* - (-1)^{k+1} [1 - \rho]}{(k+1)(1 - \rho)} \\ & \quad - \frac{\rho P_B [(B+1)^{k+1} - B^{k+1} + (-1)^{k+1}]}{(k+1)(1 - \rho)}. \end{aligned} \quad (23)$$

The correctness of equation (23) can be verified by setting $B \rightarrow \infty$, then $P_B \rightarrow 0$, equation (23) will approach to (6). Actually, from Theorem 2, we get

$$M_1^* = \frac{\rho}{1-\rho} - \frac{(B+1)\rho P_B}{1-\rho}, \quad (24)$$

$$M_2^* = \frac{(1+\rho)M_1^*}{1-\rho} - \frac{[(B+1)^3 - B^3 - 1]\rho P_B}{3(1-\rho)}, \quad (25)$$

$$M_3^* = \frac{6(1+\rho)M_2^* - 4(1-\rho)M_1^* + 2\rho}{\frac{4(1-\rho)}{[(B+1)^4 - B^4 + 1]\rho P_B}} - \frac{4(1-\rho)}{4(1-\rho)}. \quad (26)$$

Proof:

When $k \geq 1$, define $M_k^* = \sum_{n=0}^B n^k P_n$. Multiplying n^{k+1} on both sides of the second equation of (16), using the same method as in Theorem 1 yields

$$\begin{aligned} & \mu \sum_{l=0}^{k+1} \binom{k+1}{l} (-1)^l \left[\sum_{n=1}^{B-1} (n+1)^{k+1-l} P_{n+1} \right] \\ &= (\lambda + \mu) \sum_{n=1}^{B-1} n^{k+1} P_n \\ & \quad - \lambda \sum_{l=0}^{k+1} \binom{k+1}{l} \left[\sum_{n=1}^{B-1} (n-1)^{k+1-l} P_{n-1} \right], \quad (27) \\ & \mu \sum_{l=0}^{k+1} \binom{k+1}{l} (-1)^l [M_{k+1-l}^* - P_1] - (-1)^{k+1} \mu P_0 \\ &= (\lambda + \mu) [M_{k+1}^* - B^{k+1} P_B] - \\ & \quad \lambda \sum_{l=0}^{k+1} \binom{k+1}{l} [M_{k+1-l}^* - (B-1)^{k+1-l} P_{B-1} - B^{k+1-l} P_B] \quad (28) \end{aligned}$$

Combining (28) with the first and the third equations in (16), we obtain the formula to calculate M_k^* :

$$\begin{aligned} M_k^* &= \frac{\sum_{l=2}^{k+1} \binom{k+l}{l} [(-1)^l \mu + \lambda] M_{k+1-l}^* - (-1)^{k+1} [\mu - \lambda]}{(k+1)(\mu - \lambda)} \\ & \quad - \frac{\lambda P_B [(B+1)^{k+1} - B^{k+1} + (-1)^{k+1}]}{(k+1)(\mu - \lambda)} \\ &= \frac{\sum_{l=2}^{k+1} \binom{k+l}{l} [(-1)^l + \rho] M_{k+1-l}^* - (-1)^{k+1} [1 - \rho]}{(k+1)(1 - \rho)} \\ & \quad - \frac{\rho P_B [(B+1)^{k+1} - B^{k+1} + (-1)^{k+1}]}{(k+1)(1 - \rho)}. \quad (29) \end{aligned}$$

Q.E.D.

It is known that each derivative operation on a fraction like the MGF (22) will result in two new terms, if deriving the k th derivative of (22) to compute the k th moment, the computation complexity is $O(2^k)$; while from equation (29), the complexity to compute the k th moment is only $O(k^2)$.

IV. CONCLUSION

This letter provides the recursive formulas to compute the moments of queue length in the $M/M/1$ and $M/M/1/B$ queues. Our method to derive the recursive formula can also be applied to more complex queueing systems, say, the other birth-death queueing systems (e.g. $M/M/c$) or some Markovian queues [12] (e.g. $M/E_r/1$ queue), to get the recursive relationship between the moments of queue length.

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