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複雑な挙動を示すにはどれだけの非線形性が必要か？

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あらまし 本報告では、広いクラスの非線形振動システムに生じるカオス的挙動の条件をリヤプノフ指数を用いて解析する。非線形性とダンピングのトレードオフで定義されるある閾値を挙動の振幅が超えたとき、サドルノード分岐、周期倍分岐、カオスのような不安定な挙動が生じることを示す。
キーワード 散逸、非線形性、リヤプノフ指数、安定性、分岐、非線形振動子

How much nonlinearity is necessary for complex behavior?

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Abstract The analytical conditions for the appearance of chaotic motion in a broad class of nonlinear oscillatory systems are analyzed by means of Lyapunov exponents. It is shown that any instability, like, e.g., saddle-node bifurcations, period-doubling cascades or chaotic behavior can occur when the amplitude of motion exceeds certain threshold, defined by the tradeoff between nonlinearity and damping.
Keywords: dissipation, nonlinearity, Lyapunov exponents, stability, bifurcation, nonlinear oscillator

1. Introduction

Dissipation and nonlinearity are two principal factors responsible for the appearance of complex behavior in dynamical systems. The former one defines the contraction of phase space with time that ensures the existence of attracting sets (regular and strange attractors), whereas the latter produces the stretching and folding necessary for the formation of self-similar structures in the phase space, like, e.g. strange attractors or fractal basin boundaries. Despite general understanding of the interplay between these two factors in nonlinear dynamics, it remains an extremely difficult task to predict the behavior of the given system for arbitrary levels of nonlinearity and dissipation.

In this paper we analyze how the tradeoff between nonlinear and dissipative properties results in the appearance of complex behavior in passive non-autonomous oscillators, an important class of dynamical systems used in many applications in electronics, optics, acoustics, etc. [1]. As a rule, for a generic nonlinear oscillator no rigorous analytical method exists that can be used for predicting the dependence of its behavior on the control parameters. That is why a traditional scheme for understanding the dynamics of nonlinear systems can be roughly formulated as follows. At the initial stage, a linear system is introduced that can approximate the original system, thus providing some intuitive view on its properties. Then, the broader class of equations called *weakly nonlinear systems* is considered, that gives a solution in the area of controls where nonlinearity and dissipation are “small”. Such an approach allows a set of powerful analytical methods to be used based on the ideas of asymptotic expansions and/or averaging, that finally enables one to obtain constructive results concerning the stability of regular oscillatory regimes and their bifurcations. Finally, for arbitrarily large values of the parameters responsible for nonlinear and dissipative properties of the system, the extensive numerical experiments have to be utilized for the analysis of, e.g., chaotic attractors, intermittency, attractor crises, and other bifurcations typical of strongly nonlinear regimes.

It should be noted that, although the asymptotic methods can be effectively exploited for rather moderate values of nonlinearity and dissipation, the question of the validity of such results remains open and the predictions obtained by these methods should be always checked by the direct numerical simulation. This apparent drawback of the asymptotic methods originates from the absence of unambiguous definition of the notions of "weak" and "strong" nonlinearity, as well as of constructive methods of calculating the areas of "weak nonlinearity" in the space of control parameters. Intuitively it is clear that the degree of nonlinearity in the given dynamical system is related to the amplitude of motion, being large for the big amplitude and small for the motion in the close vicinity of equilibrium. A classical example of such a system is mathematical pendulum, whose dynamics is well described by linear equations in the vicinity of its equilibrium position, and become quite complicated in the area of big deviations.

The problem of defining the threshold amplitude in a nonlinear system, below which the dynamics can be considered linear or "almost" linear, remains one of the most important from the applications viewpoint. It is closely related to the problem of stability of the given system describing, e.g., an electronic device or circuit and thus constitutes a fundamental issue to be solved in the development of any application utilizing nonlinear elements [2,3].

Until very recently, the stability analysis in nonlinear dynamical systems was based on the assumption that only periodic and quasiperiodic motion can exist in the stationary regime. Under such an assumption, the final goal of predicting the stability of the given system at the given values of controls can be reached by the Floquet-type analysis [1,2] of linearized equations in the vicinity of a stationary periodic or quasiperiodic oscillation. Nowadays, with the discovery of the phenomenon of deterministic chaos, it became clear that the traditional methods of stability analysis have to be revised with taking into account the possibility for the chaotic motion to appear. On the other hand, the burst of activity in the field of nonlinear dynamics spurred by this discovery in the middle of 60-s, has led to the invention of new powerful methods of analysis that can be efficiently applied for predicting the raise of instabilities in dynamical systems of virtually any physical origin, including various electronic devices and circuits [3].

In this paper we apply a recently proposed method of stability analysis [4] based on the notion of Lyapunov exponents for predicting the general kind of instability in oscillatory systems. From a slightly different perspective, the method establishes the threshold of stability in terms of the amplitude of oscillations that guarantees the stable operation of the devices or circuits described by similar sets of differential equations.

Lyapunov characteristic exponents (LCE) provide a quantitative measure of stretching and contracting deformations of an infinitesimally small phase space sphere in the vicinity of an arbitrary trajectory in a dynamical system. So defined, they also characterize the divergence (convergence) rates of two initially close trajectories residing on an attractor and serve as indicators of the stability of motion. Total number of Lyapunov exponents that a system possesses is equal to the dimension of the phase space, or, in other words, the number of independent variables necessary to fully characterize the motion. Being invariant under a smooth change of coordinates, LCE provide a useful quantitative measure of stability for various types of motion including complex quasiperiodic orbits and chaos and, together with other dynamic invariants like fractal dimension and Kolmogorov-Sinai entropy play an important role in the theory of nonlinear dynamical systems.

2. Problem posing and mathematical formalism.

Under rather general assumptions on the type of dynamical system that models, e.g., the electric current in a simple nonlinear circuit [1] or the temporal variation of mode amplitude in a electromagnetic resonator filled with a nonlinear medium [3], the mathematical description can be given by differential equations of the form

$$\frac{d^2x_i}{dt^2}+\omega_{0i}^2x_i+\delta_i\frac{dx_i}{dt}+N_i\left(x,\frac{dx}{dt}\right)=f_i(t) \qquad (1)$$

where x_i is the generalized coordinate of i-th oscillator, $i=1,2,...,m$, number of oscillators, ω_{0i} - their natural frequencies, δ_i, damping parameters, $N_i(.)$, are nonlinear functions describing coupling between the oscillators and nonlinear properties of the system, $f_i(t)$, external perturbations. In the discussion given below, we restrict the consideration by the case of the focus type singular point at the origin of each oscillator that corresponds to the physically important case of small dissipation and coupling between oscillators.

It is evident that the set of nonlinear oscillators (1) belongs to a more general class of dynamical systems described by the equation

$$\frac{dx}{dt} = F(x,t), \qquad x \in \Re^n \qquad (2).$$

To analyze the stability of an arbitrary solution of Eq. (1) $\mathbf{x}^*(t)$, one has to consider the linearized system [1,2]

$$\frac{dy}{dt} = \hat{J}(x^*(t))y, \qquad (3)$$

where $\hat{J}(x^*(t)) \equiv \partial F(x^*(t))/\partial x$ is n×n time dependent Jacobian matrix, y is an n-vector in the tangent space corresponding to an infinitesimal perturbation of the trajectory $x^*(t)$. The standard algorithm of calculating the spectrum of LCE [5] consists in solving the equations (3) simultaneously with (2) for a set of mutually orthonormal vectors $\{y_k\}$(k = 1, 2, ..., n) and estimating the average expansion rates for the lengths $\rho_k = \|y_k\|$ of the vectors $\{y_k\}$. The general solution of the equation (3) is given by

$$y(t) = \hat{M}(t)y(0),$$

where $\hat{M}(t)$ is the fundamental matrix of solutions for the equation (3). It has been shown by Oseledec [6] that for almost any choice of initial conditions there exists the following long time limit for the initially orthonormal vectors $y_k(0)$

$$\lambda_k = \lim_{t\to\infty} \frac{1}{t} ln\left\|\hat{M}(t)y_k(0)\right\|. \qquad (4)$$

In other words this means that asymptotically, in the limit of $t \to \infty$, the evolution of $\|y_k\|$ is approximated by $\|y_k(t)\| = \|y_k(0)\|e^{\lambda_k t}$, where the exponents λ_k constitute the spectrum of LCE.

The equation (3) can be rewritten in the polar coordinate frame for the amplitude $\rho = \|y\|$ and directions φ_m $(m = 1,2,..,n-1)$ of an arbitrary vector y in the tangent space.

$$\frac{d}{dt}[ln\,\rho(t)] = P(\varphi_1, \varphi_2, ..., \varphi_{n-1}), \qquad (5a)$$

$$\frac{d\varphi_m}{dt} = \Phi(\varphi_1, \varphi_2, ..., \varphi_{n-1}), \qquad (5b)$$

where y_l are Cartesian components of the vector y, $\rho^2 = \sum_{l=1}^{n} y_l^2$; and the angles φ_m can be found by the direct formulas defining the transition from Cartesian to spherical coordinates in $\Re^n$ [4].

It is easy to show merely from the definition of the LCE that

$$\lambda_k = \lim_{T\to\infty} \frac{1}{T}\int_0^T P_k(\varphi_1(t), \varphi_2(t), ..., \varphi_{n-1}(t))dt.$$

i.e., the LCE are long time averages of corresponding functions of angular coordinates of the vectors $\{y_k\}$. The functions $P_k(\varphi_1(t), \varphi_2(t), ..., \varphi_{n-1}(t))$ depend on both the time and phase space coordinates and constitute the spectrum of instantaneous growth rates $\mu_k(t)$.

$$\mu_k(t) = \frac{d[ln\,\rho_k(t)]}{dt} = P_k(\varphi_1(t), \varphi_2(t), ..., \varphi_{n-1}(t))$$

If we arrange the values of λ_k in descending order, then the instability means the positive value of the first (largest) LCE, i.e. $\lambda_1 > 0$. It is evident, that λ_1 can take a positive value only if μ_1 can be greater than zero. On the contrary, if the inequality

$$\mu_1(t) < 0 \qquad (6)$$

holds all the time, the system is asymptotically stable, i.e. all the perturbations are exponentially shrinking with time and, hence, unstable motions are precluded. From the inequality (6), together with Eqs. (2), (3), it appears possible to obtain the relation between the control parameters and phase space coordinates which guarantees that the system is "safe" in the sense that, if the trajectory never leaves the region with negative values of μ_1, no instability appears. The goal is reached by analyzing the structure of the function $P_1(\varphi_1(t), \varphi_2(t), ..., \varphi_{n-1}(t))$, together with solutions of Eq. (2), which define the dynamics of angles φ_m through Eqs. (5). It should be however noted that a straightforward calculation of the function P_1 does not always lead to the explicit equation for the border of the asymptotic stability area in the phase space. This happens due to the presence of both the expanding and contracting directions around a typical trajectory that is a consequence of the affine character of the phase flow in the vicinity of a generic stable fixed point. Fortunately, the particular form of the function P_1 depends on the choice of coordinates, and in many cases it turns out possible to obtain the borders of the asymptotic stability area by introducing a linear change of coordinates diagonalizing the linear part of the flow $\mathbf{F}(\mathbf{x}, t)$ in the vicinity of an arbitrary point in the phase space. This kind of transformation is known to be a standard tool in the analysis of differential equations [7].

3. Non-autonomous passive nonlinear oscillator

As an example of a particular system governed by the equations of the type (1), we take the single nonlinear oscillator (motion in a potential well with a potential $U(x)$ defined by $N(x) \equiv dU(x)/dx$)

$$\frac{d^2x}{dt^2} + \delta\frac{dx}{dt} + \omega_0^2 x + \varepsilon N(x) = f(t) \qquad (7)$$

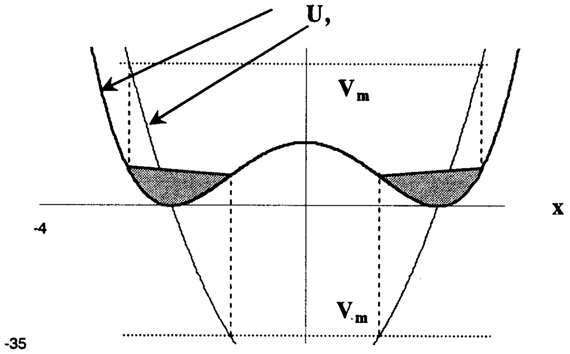

Fig. 1. Areas of asymptotic stability (shaded) for the case of two-well cubic potential.

that has been used as a basic model in many problems of mechanics, electronics, optics, electromagnetic field theory, etc.

By introducing the variables $x_1 = x; x_2 = dx/dt$ the system (7) is transformed to the standard form

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -\delta x_2 - \omega_0^2 x_1 - \varepsilon N(x_1) + f(t) \end{aligned} \qquad (8)$$

and the variational equations (3) in the vicinity of an arbitrary trajectory $x^*(t)$ for this system look like

$$\begin{aligned} \frac{dy_1}{dt} &= y_2 \\ \frac{dy_2}{dt} &= -\omega_0^2 y_1 - \delta y_2 - \varepsilon V(x^*(t)) y_1 \end{aligned} \qquad (9)$$

where $y_{1,2}$ are the components of the perturbation vector y, $V(x^*(t)) = \left.\frac{dN}{dx_1}\right|_{x_1 = x^*(t)}$. Then, after the coordinate transform recasting the linear part of Eq. (8) to the canonical (diagonal) form, the explicit expression for the growth rate μ_1 follows directly from its definition and the equation for the norm of $\|y\| = \rho$ in the polar coordinates:

$$\mu_1(t) = \frac{1}{2}\left[-\delta - \frac{2\varepsilon}{\sqrt{4\omega_0^2 - \delta^2}} V(x^*(t)) \cos(2\varphi)\right],$$

Under the assumption that φ can take any value in the interval $[0;2\pi]$ we obtain the explicit formulas for the border of the asymptotic stability area

$$V_1 < \varepsilon V(x) < V_2 \qquad (10)$$

where

$$V_{1,2} = \mp\frac{1}{2}\delta\sqrt{4\omega_0^2 - \delta^2}\,.$$

Inequalities (10) thus define the limits of variation for the function $V(x)$ and, hence, for the coordinate x of the nonlinear oscillator (7), ensuring the asymptotic stability of motion.

To make precise the ideas developed above, let's specify the nonlinear function $V(x)$ and consider, e.g., the case of single potential well Duffing oscillator, i.e. the equation (7) with

$$N(x) = x^3;\;\; V(x) \equiv \frac{dN}{dx} = 3x^2\,.$$

The direct application of the procedure described above results in the following simple formula relating the value of maximal stable amplitude of motion to the parameters of nonlinearity ε and dissipation δ

$$x_{max} = \sqrt{\frac{\delta(\delta + 2\omega_0)}{3\varepsilon}} \qquad (11)$$

The general observation immediately follows from the inequality (11) that the higher the dissipation level, the larger is the area of asymptotic stability around the origin. The parameter of nonlinearity ε has opposite effect on the stability of motion, decreasing the size if the asymptotic stability area. Therefore, it can be concluded that nonlinearity and dissipation establish a balance regulating the general stability properties of the system.

4. Two coupled oscillators

Two nonlinear oscillators with a diffusive coupling is a classical system in the oscillation theory. It constitutes a natural generalization of a single-degree-of-freedom nonlinear oscillator to a more complex dynamical system, necessary for understanding the multi-mode interactions in the spatially distributed devices, like, e.g., optical wave-guides or electronic tubes. In the simplest case of identical oscillators with cubic nonlinearity $(N(x) = \beta x^3)$ and linear coupling, the system is described by the following equations

$$\begin{aligned} &\frac{d^2x_1}{dt^2} + \omega_0^2 x_1 + \varepsilon x_1^3 + \delta\frac{dx_1}{dt} + \kappa x_2 = f_1(t) \\ &\frac{d^2x_2}{dt^2} + \omega_0^2 x_2 + \varepsilon x_2^3 + \delta\frac{dx_2}{dt} + \kappa x_1 = f_2(t) \end{aligned} \qquad (12)$$

where κ stands for the coefficient of coupling between the oscillators. The direct application of the approach given above combined with the canonical linear coordinate transform results in the following equation for the largest local expansion rate:

$$2\mu_1(t) = -\delta - \frac{3\varepsilon}{2}\left[U\left(\frac{1}{\omega_1}\sin 2\phi\cos^2\theta + \frac{1}{\omega_2}\sin 2\varphi\sin^2\theta\right)\right.$$
$$\left. + V\left(\frac{1}{\omega_1}\sin\phi\cos\varphi\sin 2\theta + \frac{1}{\omega_2}\cos\phi\sin\varphi\sin 2\theta\right)\right]$$

where $U = x_1^2 + x_2^2$; $V = x_1^2 - x_2^2$; $\omega_{1,2}^2 = \omega_0^2 - \frac{\delta^2}{4} \mp \kappa$,

$\left(\omega_0^2 - \frac{\delta^2}{4} \mp \kappa \ge 0\right)$, ϕ, φ, θ are the angles φ_m in Eq. (5).

Then, the condition of absolute stability (6) can be formulated for this system as

$$x_1^2, x_2^2 < \frac{\delta\omega_1\omega_2}{3\varepsilon(\omega_1 + \omega_2)} \qquad (13)$$

Therefore, the stability of the system of two coupled oscillators as a whole is defined by the amplitude of oscillations of each of its subsystems. Note, that, exactly like in the case of a single oscillator, the dissipation δ and nonlinearity ε cause opposite effects on the stability. However, since the partial frequencies ω_1 and ω_2 also depend on δ, as well as on the parameter of coupling κ, the overall effect of dissipation on the system stability becomes more complicated.

5. Conclusion

In this paper we discuss a novel method for the stability analysis of oscillatory systems based on the notions of asymptotic stability and Lyapunov exponents. Our approach allows defining the area in the phase space where all trajectories are asymptotically stable and, therefore, no bifurcation typical for nonlinear systems can occur. In fact, the technique allows obtaining a threshold in the amplitude of motion that ensures the asymptotic stability of motion restricted from above by this value. The maximal stable amplitude is shown to be conditioned by the tradeoff between nonlinear and dissipative properties of the system. It is shown that cumulative effect of all the control parameters on the stability of the system can be expressed via amplitude of motion in the phase space. In particular, it is demonstrated that the dissipation level as well as the parameters regulating the nonlinear properties control the amount of nonlinearity necessary for the appearance of complex behavior in nonlinear systems.

It is interesting to note, that the found out area of asymptotic stability in the phase space can be used for specifying the notion of a weakly non-linear behavior of the system. As we already noted in the introduction, one of the fundamental problems in the theory of nonlinear oscillators is to define the values of control parameters and amplitude of motion, where the behavior of the system can be considered almost linear. The method described above not only establishes the required relation between the controls and phase space coordinates where strongly nonlinear effects are absent, but also provides a constructive way of defining the very notion of "linearity" and allows to quantify the difference between "linear" and "nonlinear" behaviors. Since the area of asymptotic stability guarantees the absence of any strongly nonlinear effects (bifurcations) inside of it; then any asymptotic methods of analysis can be successfully applied here for finding the approximate solutions.

It has been recently recognized that in many oscillatory systems the threshold of instability may be strongly dependent on the frequency content of the external signal. As it was shown, e.g., in [8], the change of harmonic to bifrequency excitation in an equation of class (7) results in considerable lowering of the instability onset in the intensity of the external force. A natural question stems from these findings: what is the lowest possible level of excitation that can result in a destabilization of the system? As we have demonstrated with several examples of nonlinear oscillators, the analysis of asymptotic stability in terms of LCE allows answering this question and estimating the maximal stable amplitude of motion, and thus provides a necessary condition for any bifurcation to occur. We would like to stress that the method we propose is independent from the type of external force and dimensionality of the dynamical system, therefore, it yields a fundamental limit for all types of instabilities, including chaotic motion, to appear in a broad class of nonlinear circuits and systems.

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