PAPER

Approximation and Analysis of Non-linear Equations in a Moment Vector Space

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SUMMARY Moment vector equations (MVEs) are presented for use in approximating and analyzing multi-dimensional non-linear discrete- and continuous-time equations. A non-linear equation is expanded into simultaneous equations of generalized moments and then reduced to an MVE of a coefficient matrix and a moment vector. The MVE can be used to analyze the statistical properties, such as the mean, variance, covariance, and power spectrum, of the non-linear equation. Moreover, we can approximately express a combination of non-linear equations by using a combination of MVEs of the equations. Evaluation of the statistical properties of Lorenz equations and of a combination of logistic equations based on the MVE approach showed that MVEs can be used to approximate non-linear equations in statistical measurements.

key words: *approximation, linearization, non-linear, statistics*

1. Introduction

Many everyday systems are large, complicated, and non-linear. They often comprise a large number of small systems that interact in a complicated manner. To design and control these systems, we need to know their various properties. However, even if we knew all the properties of the elements that constitute the target system, we cannot always control the elements. We may find it more useful to know the macroscopic and statistical properties of the target system, such as the moments, response time, and power spectra, than the microscopic and minute ones.

Various methods have been developed for approximately analyzing non-linear systems, including the Galerkin method and ones based on perturbation analysis and asymptotic analysis. These methods expand the solution into a series and determine the coefficient of each function. By using these methods, we can obtain approximate solutions to deterministic differential equations [1], [2], [3], stochastic differential equations [4], and partial differential equations of a probability density function [5]. A linearization approach that creates a linear model by linearizing the target system at the equilibrium point is widely used. With this model, we can use linear control theory [6] to control the target system.

However, approximate solutions obtained by the above methods are rather complex, or the linear model obtained in the neighborhood of the equilibrium point is too simple to express global properties. Thus, we cannot always use these methods to control everyday systems or to investigate their macroscopic and statistical properties. Moment vector equations (MVEs), which were recently developed [7], can be used to analyze the statistical properties, such as the mean, variance, covariance, and power spectrum, for one-dimensional discrete-time systems. However, this does not mean that MVEs always approximate the non-linear systems or that we can use MVEs as an approximation of non-linear systems to control them. Moreover, MVEs should also work for multi-dimensional or continuous-time systems.

The systematic procedures presented in this paper solve these problems by expanding a previous work on one-dimensional discrete-time systems [7]. Using these procedures, we can construct MVEs and use them to analyze the statistical properties of multi-dimensional non-linear discrete- and continuous-time equations. A combination of non-linear equations can be expressed approximately by using a combination of MVEs of the non-linear equations, demonstrating that an MVE approximates a non-linear system itself. Evaluation of the statistical properties of Lorenz equations and those of a combination of logistic equations by using MVEs showed that MVEs can be used as an approximation of non-linear equations in statistical measurements.

2. MVEs for Multi-dimensional Systems

This section presents systematic procedures by expanding a previous work [7] to approximate various non-linear equations to MVEs.

2.1 MVEs for Discrete-time Systems

Let us consider the following multi-dimensional discrete-time non-linear systems

$$s_\ell(n+1) = f_\ell(\boldsymbol{s}(n)) \quad \text{for} \quad 1 \le \ell \le L, \tag{1}$$

where $\boldsymbol{s} \overset{\text{def}}{=} {}^{\mathrm{t}}(s_1,\cdots,s_L)$, L is the number of variables, $n = 0, 1, 2, \cdots$ is a discrete time, $\boldsymbol{S} \overset{\text{def}}{=} \{\boldsymbol{s} | \check{s}_\ell \le s_\ell \le \check{s}_\ell + T_\ell \text{ for } 1 \le \ell \le L\}$ is the domain of the definition of $\boldsymbol{s}(n)$, and t denotes transposition.

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Let $\{\phi_i\}$ be a linear independent system that constitutes the basis of $\boldsymbol{S}$ (see Appendix A), and let us abbreviate the variables defined above as follows:

$$\begin{aligned} s_\ell &\stackrel{\text{def}}{=} s_\ell(n), \\ s'_\ell &\stackrel{\text{def}}{=} s_\ell(n+1), \\ \boldsymbol{s} &\stackrel{\text{def}}{=} \boldsymbol{s}(n), \\ \phi_i &\stackrel{\text{def}}{=} \phi_i(\boldsymbol{s}(n)), \\ \phi'_i &\stackrel{\text{def}}{=} \phi_i(\boldsymbol{s}(n+1)). \end{aligned} \tag{2}$$

To derive an MVE, let us assume the following with respect to Eq. (1):

Assumption 1: We can expand $E[\phi'_i|\boldsymbol{s}]$ in a Fourier series as follows:

$$E[\phi'_i|\boldsymbol{s}] = \sum_{j=0}^{N} a_{ij}\phi_j + \varepsilon_i(\boldsymbol{s}), \tag{3}$$

where $E[\cdot]$ is a mathematical expectation, $\varepsilon_i(\boldsymbol{s})$ is a residual, and ϕ_0 is a constant.

By using Eq. (3), we can expand $E[\phi'_i]$ as follows:

$$\begin{aligned} E[\phi'_i] &= \int \phi'_i p(\phi'_i)\mathrm{d}\phi'_i \\ &= \int \phi'_i \int p(\boldsymbol{s})p(\phi'_i|\boldsymbol{s})\mathrm{d}\boldsymbol{s}\mathrm{d}\phi'_i \\ &= \int p(\boldsymbol{s})E[\phi'_i|\boldsymbol{s}]\mathrm{d}\boldsymbol{s} \\ &= \sum_{j=0}^{N} a_{ij}E[\phi_j] + E[\varepsilon_i(\boldsymbol{s})], \end{aligned} \tag{4}$$

where $p(\cdot)$ denotes a probability density function. When $\{\phi_i\}$ is an orthonormal basis, a_{ij} is obtained by using Eq. (A$\cdot$2) as follows:

$$a_{ij} = \int_{\boldsymbol{S}} \phi_i(\boldsymbol{f}(\boldsymbol{s}))\phi_j(\boldsymbol{s})\mathrm{d}\boldsymbol{s}, \tag{5}$$

where $\boldsymbol{f} \stackrel{\text{def}}{=} {}^{\text{t}}(f_1, \cdots, f_L)$. Let us assume that $E[\varepsilon_i(\boldsymbol{s})] = 0$, and let us define $\tilde{\boldsymbol{x}}(n)$ and $\tilde{\boldsymbol{A}}$ as follows:

$$\tilde{\boldsymbol{x}}(n) \stackrel{\text{def}}{=} {}^{\text{t}}(E[\phi_0(\boldsymbol{s}(n))], \cdots, E[\phi_N(\boldsymbol{s}(n))]),$$

$$\tilde{\boldsymbol{A}} \stackrel{\text{def}}{=} \begin{bmatrix} a_{00} & \cdots & a_{0N} \\ \vdots & \ddots & \vdots \\ a_{N0} & \cdots & a_{NN} \end{bmatrix}.$$

Then Eq. (4) is expressed by the following MVE:

$$\tilde{\boldsymbol{x}}(n+1) = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}(n). \tag{6}$$

In Assumption 1, we assumed that $E[\phi_0(\boldsymbol{s}(n))]$ is constant ϕ_0. Thus, by rewriting Eq. (6), we obtain an MVE, which is a linear equation in a moment vector space, as follows:

$$\boldsymbol{x}(n+1) = \boldsymbol{A}\boldsymbol{x}(n) + \boldsymbol{B}, \tag{7}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{x}(n)$ are defined as follows:

$$\boldsymbol{x}(n) \stackrel{\text{def}}{=} {}^{\text{t}}(E[\phi_1(\boldsymbol{s}(n))], \cdots, E[\phi_N(\boldsymbol{s}(n))]),$$

$$\boldsymbol{A} \stackrel{\text{def}}{=} \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{bmatrix}, \boldsymbol{B} \stackrel{\text{def}}{=} \begin{bmatrix} a_{10}\phi_0 \\ \vdots \\ a_{N0}\phi_0 \end{bmatrix}.$$

2.2 MVEs for Continuous-time Systems

In this section, MVEs for multi-dimensional continuous-time non-linear systems are derived in the same manner as in Sect. 2.1. Let us consider the following systems

$$\dot{s}_\ell(t) = f_\ell(\boldsymbol{s}(t)) \quad \text{for} \quad 1 \le \ell \le L, \tag{8}$$

where t is a continuous time, $\dot{s}_\ell(t)$ denotes $\mathrm{d}s_\ell(t)/\mathrm{d}t$, and $\boldsymbol{s}(t) \in \boldsymbol{S}$.

Because $E[\cdot]$ is a linear operator, $\mathrm{d}E[\phi_i(\boldsymbol{s}(t))]/\mathrm{d}t = E[\mathrm{d}\phi_i(\boldsymbol{s}(t))/\mathrm{d}t]$. Thus, by using the following abbreviation instead of Eq. (2)

$$\begin{aligned} s_\ell &\stackrel{\text{def}}{=} s_\ell(t), \\ s'_\ell &\stackrel{\text{def}}{=} \mathrm{d}s_\ell/\mathrm{d}t, \\ \boldsymbol{s} &\stackrel{\text{def}}{=} \boldsymbol{s}(t), \\ \phi_i &\stackrel{\text{def}}{=} \phi_i(\boldsymbol{s}(t)), \\ \phi'_i &\stackrel{\text{def}}{=} \mathrm{d}\phi_i(\boldsymbol{s}(t))/\mathrm{d}t, \end{aligned} \tag{9}$$

and assuming that Assumption 1 holds, we can obtain the following equation from Eq. (4):

$$\begin{aligned} \mathrm{d}E[\phi_i]/\mathrm{d}t &= E[\phi'_i] \\ &= \sum_{j=0}^{N} a_{ij}E[\phi_j] + E[\varepsilon_i(\boldsymbol{s})]. \end{aligned} \tag{10}$$

Here, by using Eq. (8), we can express ϕ'_i as follows:

$$\phi'_i = \sum_{\ell=1}^{L} \frac{\partial \phi_i}{\partial s_\ell} f_\ell(\boldsymbol{s}).$$

Thus, when $\{\phi_i\}$ is an orthonormal basis, a_{ij} is obtained by using Eq. (A$\cdot$2) as follows:

$$a_{ij} = \int_{\boldsymbol{S}} (\sum_{\ell=1}^{L} \frac{\partial \phi_i(\boldsymbol{s})}{\partial s_\ell} f_\ell(\boldsymbol{s}))\phi_j(\boldsymbol{s})\mathrm{d}\boldsymbol{s}. \tag{11}$$

Let us assume that $E[\varepsilon_i(\boldsymbol{s})] = 0$, and let us define $\tilde{\boldsymbol{x}}(t)$ by

$$\tilde{\boldsymbol{x}}(t) \stackrel{\text{def}}{=} {}^{\text{t}}(E[\phi_0(\boldsymbol{s}(t))], \cdots, E[\phi_N(\boldsymbol{s}(t))]).$$

From Eq. (10), we obtain the following MVE:

$$\dot{\tilde{\boldsymbol{x}}}(t) = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}(t), \tag{12}$$

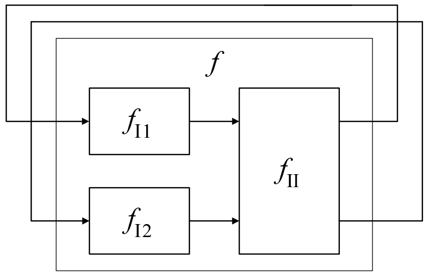

Fig. 1 A combination of three discrete-time systems.

where $\dot{\tilde{\boldsymbol{x}}}(t) \stackrel{\text{def}}{=} \mathrm{d}\tilde{\boldsymbol{x}}(t)/\mathrm{d}t$. As in Sect. 2.1, we assume that $E[\phi_0(\boldsymbol{s}(t))]$ is constant ϕ_0. Thus, the above equation can be rewritten as the following MVE:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}, \tag{13}$$

where $\dot{\boldsymbol{x}}(t) \stackrel{\text{def}}{=} \mathrm{d}\boldsymbol{x}(t)/\mathrm{d}t$, and $\boldsymbol{x}(t)$ is defined by

$$\boldsymbol{x}(t) \stackrel{\text{def}}{=} {}^{\mathrm{t}}(E[\phi_1(\boldsymbol{s}(t))], \cdots, E[\phi_N(\boldsymbol{s}(t))]).$$

2.3 A Combination of Discrete-time Systems

Let us consider the following three discrete-time systems in order to explain how we can express a combination of discrete-time systems:

$$\begin{aligned} \boldsymbol{s}_{\mathrm{I}\ell}(n+1) &= \boldsymbol{f}_{\mathrm{I}\ell}(\boldsymbol{s}_{\mathrm{I}\ell}(n)) \quad \text{for} \quad \ell = 1, 2, \\ \boldsymbol{s}_{\mathrm{II}}(n+1) &= \boldsymbol{f}_{\mathrm{II}}(\boldsymbol{s}_{\mathrm{II}}(n)). \end{aligned}$$

By using bases $\{\phi_i\}$ and $\{\phi_{\ell i}\}$, we can express the MVEs of $\boldsymbol{f}_{\mathrm{I1}}$, $\boldsymbol{f}_{\mathrm{I2}}$, and $\boldsymbol{f}_{\mathrm{II}}$ by

$$\begin{aligned} &\tilde{\boldsymbol{x}}_{\mathrm{I}\ell}(n+1) = \tilde{\boldsymbol{A}}_{\mathrm{I}\ell}\tilde{\boldsymbol{x}}_{\mathrm{I}\ell}(n) \quad \text{for} \quad \ell = 1, 2, \\ &\tilde{\boldsymbol{x}}_{\mathrm{II}}(n+1) = \tilde{\boldsymbol{A}}_{\mathrm{II}}\tilde{\boldsymbol{x}}_{\mathrm{II}}(n), \end{aligned}$$

where

$$\begin{aligned} &\tilde{\boldsymbol{x}}_{\mathrm{I}\ell}(n) \stackrel{\text{def}}{=} {}^{\mathrm{t}}(E[\phi_{\ell 0}(\boldsymbol{s}_{\mathrm{I}\ell}(n))], \cdots, E[\phi_{\ell N_\ell}(\boldsymbol{s}_{\mathrm{I}\ell}(n))]), \\ &\tilde{\boldsymbol{x}}_{\mathrm{II}}(n) \stackrel{\text{def}}{=} {}^{\mathrm{t}}(E[\phi_0(\boldsymbol{s}_{\mathrm{II}}(n))], \cdots, E[\phi_N(\boldsymbol{s}_{\mathrm{II}}(n))]), \\ &\tilde{\boldsymbol{\phi}} \stackrel{\text{def}}{=} {}^{\mathrm{t}}(\phi_0, \cdots, \phi_N), \\ &\tilde{\boldsymbol{\phi}}_\ell \stackrel{\text{def}}{=} {}^{\mathrm{t}}(\phi_{\ell 0}, \cdots, \phi_{\ell N_\ell}). \end{aligned}$$

Let us now construct a new system, $\boldsymbol{f}$, by connecting the above three systems, $\boldsymbol{f}_{\mathrm{I1}}$, $\boldsymbol{f}_{\mathrm{I2}}$, and $\boldsymbol{f}_{\mathrm{II}}$, as shown in Fig. 1. That is,

$$\begin{aligned} &\boldsymbol{s}(n+1) = \boldsymbol{f}(\boldsymbol{s}(n)), \\ &\boldsymbol{s}(n+1) \stackrel{\text{def}}{=} \boldsymbol{s}_{\mathrm{II}}(n+1), \\ &\boldsymbol{s}(n) \stackrel{\text{def}}{=} {}^{\mathrm{t}}({}^{\mathrm{t}}\boldsymbol{s}_{\mathrm{I1}}(n), {}^{\mathrm{t}}\boldsymbol{s}_{\mathrm{I2}}(n)), \\ &\boldsymbol{f} \stackrel{\text{def}}{=} \boldsymbol{f}_{\mathrm{II}}(\boldsymbol{f}_{\mathrm{I1}}, \boldsymbol{f}_{\mathrm{I2}}). \end{aligned} \tag{14}$$

For given $\tilde{\boldsymbol{\phi}}_\ell$, let $\tilde{\boldsymbol{\phi}}$ and $\tilde{\boldsymbol{x}}(n)$ be

$$\tilde{\boldsymbol{x}}(n) \stackrel{\text{def}}{=} {}^{\mathrm{t}}(E[\phi_0(\boldsymbol{s}(n))], \cdots, E[\phi_N(\boldsymbol{s}(n))]),$$

$$\tilde{\boldsymbol{\phi}} \stackrel{\text{def}}{=} \tilde{\boldsymbol{\phi}}_1 \otimes \tilde{\boldsymbol{\phi}}_2, \tag{15}$$

and let us express the MVE of $\boldsymbol{f}$ by

$$\tilde{\boldsymbol{x}}(n+1) = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}(n), \tag{16}$$

where $\otimes$ denotes a matrix direct product. From Eqs. (14) and (15), the following equation holds:

$$\tilde{\boldsymbol{x}}(n) = \tilde{\boldsymbol{x}}_{\mathrm{I1}}(n) \otimes \tilde{\boldsymbol{x}}_{\mathrm{I2}}(n). \tag{17}$$

Because $\boldsymbol{f}_{\mathrm{I1}}$ and $\boldsymbol{f}_{\mathrm{I2}}$ are independent, by using the following new variables

$$\begin{aligned} &\tilde{\boldsymbol{x}}'_{\mathrm{I1}} \stackrel{\text{def}}{=} \tilde{\boldsymbol{A}}_{\mathrm{I1}}\tilde{\boldsymbol{x}}_{\mathrm{I1}}(n), \\ &\tilde{\boldsymbol{x}}'_{\mathrm{I2}} \stackrel{\text{def}}{=} \tilde{\boldsymbol{A}}_{\mathrm{I2}}\tilde{\boldsymbol{x}}_{\mathrm{I2}}(n), \end{aligned}$$

the input to $\boldsymbol{f}_{\mathrm{II}}$ can be expressed by $\tilde{\boldsymbol{x}}'_{\mathrm{I1}} \otimes \tilde{\boldsymbol{x}}'_{\mathrm{I2}}$. Therefore, by using the following formulae with respect to the matrix direct product [9]

$$ {}^{\mathrm{t}}(\boldsymbol{A} \otimes \boldsymbol{B}) = {}^{\mathrm{t}}\boldsymbol{A} \otimes {}^{\mathrm{t}}\boldsymbol{B}, \tag{18}$$

$$(\boldsymbol{A} \otimes \boldsymbol{C})(\boldsymbol{B} \otimes \boldsymbol{D}) = (\boldsymbol{A}\boldsymbol{B}) \otimes (\boldsymbol{C}\boldsymbol{D}), \tag{19}$$

we obtain the following equation:

$$\begin{aligned} \tilde{\boldsymbol{x}}(n+1) &= \tilde{\boldsymbol{A}}_{\mathrm{II}}(\tilde{\boldsymbol{x}}'_{\mathrm{I1}} \otimes \tilde{\boldsymbol{x}}'_{\mathrm{I2}}) \\ &= \tilde{\boldsymbol{A}}_{\mathrm{II}}(\tilde{\boldsymbol{A}}_{\mathrm{I1}} \otimes \tilde{\boldsymbol{A}}_{\mathrm{I2}})\tilde{\boldsymbol{x}}(n). \end{aligned}$$

Therefore, coefficient matrix $\tilde{\boldsymbol{A}}$ in the MVE of $\boldsymbol{f}$ is expressed by the combination of coefficient matrices in the MVEs of $\boldsymbol{f}_{\mathrm{I1}}$, $\boldsymbol{f}_{\mathrm{I2}}$, and $\boldsymbol{f}_{\mathrm{II}}$ as follows:

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}_{\mathrm{II}}(\tilde{\boldsymbol{A}}_{\mathrm{I1}} \otimes \tilde{\boldsymbol{A}}_{\mathrm{I2}}). \tag{20}$$

Because it is assumed that $E[\phi_0(\boldsymbol{s}(n))]$ is constant ϕ_0, Eq. (16) can be rewritten as Eq. (7) in the same manner as Eq. (6).

3. Analysis

Let λ_i be the ith eigenvalue of matrix $\boldsymbol{A}$ in the MVE of Eq. (7) or (13), $\boldsymbol{e}_i$ be the eigenvector for λ_i, $\Lambda \stackrel{\text{def}}{=} \mathrm{diag}[\lambda_1, \cdots, \lambda_N]$, and $\boldsymbol{M} \stackrel{\text{def}}{=} [\boldsymbol{e}_1, \cdots, \boldsymbol{e}_N]$. In this section, various statistical properties of non-linear equations are derived based on $\boldsymbol{A}$, Λ, and $\boldsymbol{M}$ by expanding a previous work [7]. To evaluate in Sect. 4 the accuracy of the statistical properties that are obtained based on MVEs, the properties based on a numerical solution of non-linear equations are also defined in this section.

3.1 Moments for Discrete Time Systems

In this section, moments of $s_\ell(n)$ in Eq. (1) are derived under the following assumption:

Assumption 2: Equation (7), which is an MVE of Eq. (1), has a unique equilibrium point, and it does not diverge. That is, $\forall\lambda_i \neq 1$, and $\forall|\lambda_i| \leq 1$ [8].

3.1.1 Moments Based on a Numerical Solution

From the time average of the sequence of $s_\ell(n)$, $s_\ell(n+1)$, $\cdots$ obtained from the numerical solution of Eq. (1), moment $\langle\langle s_\ell(n)^k\rangle\rangle$ and covariance $\langle\langle s_\ell(n)s_\nu(n)\rangle\rangle$ based on a numerical solution are computed, where $\langle\langle\cdot\rangle\rangle$ denotes a finite-time average, and it is defined for a discrete-time variable $x(n)$ as follows:

$$\langle\langle x(n)\rangle\rangle \overset{\text{def}}{=} \frac{1}{\tau_{\max}} \sum_{\tau=0}^{\tau_{\max}-1} x(n+\tau). \tag{21}$$

Variance ${\sigma_\ell}^2$ and correlation coefficient $\rho_{\ell\nu}$ are obtained by the following equations:

$$\sigma_\ell^2 = \langle\langle s_\ell(n)^2\rangle\rangle - \langle\langle s_\ell(n)\rangle\rangle^2, \tag{22}$$

$$\rho_{\ell\nu} = \frac{\langle\langle s_\ell(n)s_\nu(n)\rangle\rangle - \langle\langle s_\ell(n)\rangle\rangle\langle\langle s_\nu(n)\rangle\rangle}{\sigma_\ell\sigma_v}. \tag{23}$$

3.1.2 Moments Based on an MVE

Let infinite-time average $\langle x(n)\rangle$ of discrete-time variable $x(n)$ be [10]

$$\langle x(n)\rangle \overset{\text{def}}{=} \lim_{\tau_{\max}\to\infty} \frac{1}{\tau_{\max}} \sum_{\tau=0}^{\tau_{\max}-1} x(n+\tau). \tag{24}$$

From Appendix B.1, $\bar{\boldsymbol{x}} \overset{\text{def}}{=} \lim_{n\to\infty}\langle \boldsymbol{x}(n)\rangle$ † is equal to $\boldsymbol{x}^*$ in Eq. (7), where $\boldsymbol{x}^*$ is the equilibrium point of $\boldsymbol{x}(n)$ in Eq. (7). Thus, when Assumption 2 holds, $(\boldsymbol{I}-\boldsymbol{A})^{-1}$ exists, and $\bar{\boldsymbol{x}}$ is derived by the following equation [8]:

$$\begin{aligned}\bar{\boldsymbol{x}} &= \boldsymbol{x}^* \\ &= (\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B},\end{aligned} \tag{25}$$

where $\boldsymbol{I}$ is a unit matrix.

Let us now expand $E[s_\ell(n+1)]$ and $E[s_\ell(n+1)s_\nu(n+1)]$ in a Fourier series as follows in the same manner as in Eq. (4):

$$\begin{aligned}E[s_\ell(n+1)] &= E[f_\ell(\boldsymbol{s}(n))] \\ &= \sum_{j=0}^{N} \zeta_{\ell;j} E[\phi_j(\boldsymbol{s}(n))],\end{aligned} \tag{26}$$

$$\begin{aligned}E[s_\ell(n+1)s_\nu(n+1)] &= E[f_\ell(\boldsymbol{s}(n))f_\nu(\boldsymbol{s}(n))] \\ &= \sum_{j=0}^{N} \zeta_{\ell\nu;j} E[\phi_j(\boldsymbol{s}(n))].\end{aligned} \tag{27}$$

At the equilibrium point of Eq. (7), $E[\phi_i(\boldsymbol{s}(n+1))] = E[\phi_i(\boldsymbol{s}(n))]$, and thus, Eqs. (26) and (27) yield $E[s_\ell(n+1)] = E[s_\ell(n)]$ and $E[s_\ell(n+1)s_\nu(n+1)] = E[s_\ell(n)s_\nu(n)]$. Let these values be $E^*[\phi_i(\boldsymbol{s})]$, $E^*[s_\ell]$, and $E^*[s_\ell s_\nu]$, respectively. From Eqs. (25) through (27), moment $\overline{E[s_\ell^k]} \overset{\text{def}}{=} \lim_{n\to\infty}\langle E[s_\ell(n)^k]\rangle$ for $k=1,2$ and covariance $\overline{E[s_\ell s_\nu]} \overset{\text{def}}{=} \lim_{n\to\infty}\langle E[s_\ell(n)s_\nu(n)]\rangle$ based on the MVE of Eq. (7) are computed by using the following equations:

$$\begin{aligned}\overline{E[s_\ell]} &= E^*[s_\ell] \\ &= \sum_{j=0}^{N} \zeta_{\ell;j} E^*[\phi_j(\boldsymbol{s})],\end{aligned} \tag{28}$$

$$\begin{aligned}\overline{E[s_\ell s_\nu]} &= E^*[s_\ell s_\nu] \\ &= \sum_{j=0}^{N} \zeta_{\ell\nu;j} E^*[\phi_j(\boldsymbol{s})].\end{aligned} \tag{29}$$

Here, $E^*[\phi_j(\boldsymbol{s})]$ is an element of $\boldsymbol{x}^*$. The variance and correlation coefficient are obtained in the same manner as in Eqs. (22) and (23).

3.2 Power Spectra for Discrete-time Systems

Periodogram [10], which is defined based on a finite-duration Fourier transform and is an estimation of the power spectrum, of $s_\ell(n)$ in Eq. (1) is derived in this section, assuming that Assumption 2 holds.

3.2.1 A Periodogram Based on a Numerical Solution

Let $F_\ell(k)$ be the time average of the Fourier transform of sequence $s_\ell(n)$, $s_\ell(n+1)$, $\cdots$ obtained from the numerical solution of Eq. (1) as follows:

$$F_\ell(k) \overset{\text{def}}{=} \langle\langle \sum_{m=0}^{W-1} s_\ell(n+m)e^{-\imath\omega_0 mk}\rangle\rangle, \tag{30}$$

where $\imath$ is an imaginary unit, $k \in \{0,1,\cdots,W-1\}$, and $\omega_0 \overset{\text{def}}{=} 2\pi/W$. From the above equation, we obtain periodogram $S_{\ell\ell}(k)$ based on the numerical solution of Eq. (1) as follows:

$$S_{\ell\ell}(k) \overset{\text{def}}{=} \frac{1}{W}|F_\ell(k)|^2. \tag{31}$$

3.2.2 A Periodogram Based on an MVE

Correlation function $r_{\ell\nu}(m)$ of $s_\ell(n)$ and $s_\nu(n)$ is defined by

$$r_{\ell\nu}(m) \overset{\text{def}}{=} \lim_{n\to\infty}\langle E[s_\ell(n)s_\nu(n+m)]\rangle. \tag{32}$$

The right-hand side of Eq. (32) is derived from Eq. (35) described in Section 3.2.3. From Eq. (32), the power spectrum of $s_\ell(n)$ based on the MVE of Eq. (1), $\hat{S}_{\ell\ell}(k)$, is computed by the following equation for $\nu=\ell$:

†The reason we have to consider $\lim_{n\to\infty}\langle \boldsymbol{x}(n)\rangle$ is explained in Appendix B.

$$\hat{S}_{\ell\nu}(k) = \sum_{m=0}^{W-1} r_{\ell\nu}(m) e^{-\imath\omega_0 mk}. \quad (33)$$

For $\nu \neq \ell$, the above equation denotes the cross power spectrum of $s_\ell(n)$ and $s_\nu(n)$ [10].

3.2.3 The MVE of the Correlation Function

The right-hand side of Eq. (32) is derived in this section. Consider the following simultaneous equations of Eqs. (7) and (26):

$$\begin{bmatrix} E[s_\nu(n+1)] \\ E[\phi_1(\boldsymbol{s}(n+1))] \\ \vdots \\ E[\phi_N(\boldsymbol{s}(n+1))] \end{bmatrix} = \hat{\boldsymbol{A}}_\nu \begin{bmatrix} E[s_\nu(n)] \\ E[\phi_1(\boldsymbol{s}(n))] \\ \vdots \\ E[\phi_N(\boldsymbol{s}(n))] \end{bmatrix} + \hat{\boldsymbol{B}}_\nu,$$

where

$$\hat{\boldsymbol{A}}_\nu \stackrel{\text{def}}{=} \begin{bmatrix} 0 & \zeta_{\nu;1} & \cdots & \zeta_{\nu;N} \\ 0 & a_{11} & \cdots & a_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{N1} & \cdots & a_{NN} \end{bmatrix}, \hat{\boldsymbol{B}}_\nu \stackrel{\text{def}}{=} \begin{bmatrix} \zeta_{\nu;0}\phi_0 \\ a_{10}\phi_0 \\ \vdots \\ a_{N0}\phi_0 \end{bmatrix}.$$

Let $\hat{\boldsymbol{x}}_{\ell\nu}(m;n)$ be

$$\hat{\boldsymbol{x}}_{\ell\nu}(m;n) \stackrel{\text{def}}{=} \begin{bmatrix} E[s_\ell(n)s_\nu(n+m)] \\ E[s_\ell(n)\phi_1(n+m)] \\ \vdots \\ E[s_\ell(n)\phi_N(n+m)] \end{bmatrix}.$$

By replacing $g_\ell(\boldsymbol{s}(n))$ with $s_\ell(n)$ in Eq. (A·13) in Appendix C, we obtain the following equation:

$$\hat{\boldsymbol{x}}_{\ell\nu}(m+1;n) = \hat{\boldsymbol{A}}_\nu \hat{\boldsymbol{x}}_{\ell\nu}(m;n) + E[s_\ell(n)]\hat{\boldsymbol{B}}_\nu. \quad (34)$$

To eliminate the effect of the initial value of Eq. (1) and to derive the power spectrum even when Eq. (34) oscillates †, consider $\bar{\hat{\boldsymbol{x}}}_{\ell\nu}(m) \stackrel{\text{def}}{=} \lim_{n\to\infty} \langle \hat{\boldsymbol{x}}_{\ell\nu}(m;n) \rangle$. We obtain $\bar{\hat{\boldsymbol{x}}}_{\ell\nu}(m)$ from the limit of the time average of Eq. (34) for $n \to \infty$ as follows:

$$\bar{\hat{\boldsymbol{x}}}_{\ell\nu}(m+1) = \hat{\boldsymbol{A}}_\nu \bar{\hat{\boldsymbol{x}}}_{\ell\nu}(m) + E^*[s_\ell]\hat{\boldsymbol{B}}_\nu. \quad (35)$$

By computing the above equation for $m = 0, 1, 2, \cdots$ step by step, we can obtain the right-hand side of Eq. (32). The initial value of the above equation, $\bar{\hat{\boldsymbol{x}}}_{\ell\nu}(0)$, is derived in Appendix D.1.

3.3 Moments for Continuous-time Systems

This section shows the moments of $s_\ell(t)$ in Eq. (8) in the same manner as in Sect. 3.1 under the following assumpiton:

Assumption 3: Equation (13), which is an MVE of Eq. (8), has a unique equilibrium point, and it does not diverge. That is, $\forall\lambda_i \neq 0$, and $\forall\text{Re}[\lambda_i] \leq 0$ [8].

3.3.1 Moments Based on a Numerical Solution

Moment $\langle\langle s_\ell(t)^k \rangle\rangle$ and covariance $\langle\langle s_\ell(t)s_\nu(t)\rangle\rangle$ are derived based on a numerical solution by solving Eq. (8) numerically. Here, $\langle\langle\cdot\rangle\rangle$ is a finite time average, and it is defined for continuous variable $x(t)$ as follows:

$$\langle\langle x(t)\rangle\rangle \stackrel{\text{def}}{=} \frac{1}{\tau_{\max}} \int_0^{\tau_{\max}} x(t+\tau)\mathrm{d}\tau. \quad (36)$$

The variance and correlation coefficient are derived in the same manner as in Eqs. (22) and (23).

3.3.2 Moments Based on an MVE

Let $\langle x(t)\rangle$ be the infinite-time average of continuous variable $x(t)$ as follows:

$$\langle x(t)\rangle \stackrel{\text{def}}{=} \lim_{\tau_{\max}\to\infty} \frac{1}{\tau_{\max}} \int_0^{\tau_{\max}} x(t+\tau)\mathrm{d}\tau. \quad (37)$$

Appendix B.2 shows that $\bar{\boldsymbol{x}} \stackrel{\text{def}}{=} \lim_{n\to\infty} \langle \boldsymbol{x}(t)\rangle$ is equal to equilibrium point $\boldsymbol{x}^*$ in Eq. (13). Thus, if Assumption 3 holds, $\boldsymbol{A}^{-1}$ exists [8], and $\bar{\boldsymbol{x}}$ is expressed by

$$\begin{aligned} \bar{\boldsymbol{x}} &= \boldsymbol{x}^* \\ &= -\boldsymbol{A}^{-1}\boldsymbol{B}. \end{aligned} \quad (38)$$

Let us expand $E[s_\ell(t)]$ and $E[s_\ell(t)s_\nu(t)]$ as follows:

$$E[s_\ell(t)] = \sum_{j=0}^{N} \eta_{\ell;j} E[\phi_j(\boldsymbol{s}(t))], \quad (39)$$

$$E[s_\ell(t)s_\nu(t)] = \sum_{j=0}^{N} \eta_{\ell\nu;j} E[\phi_j(\boldsymbol{s}(t))]. \quad (40)$$

Because $E[\phi_i(\boldsymbol{s})]$ is a constant at the equilibrium point, Eqs. (39) and (40) show that $E[s_\ell(t)]$ and $E[s_\ell(t)s_\nu(t)]$ are also constants at the equilibrium point. Let each constant be $E^*[\phi_i(\boldsymbol{s})]$, $E^*[s_\ell]$, and $E^*[s_\ell s_\nu]$, respectively. From Eqs. (38) through (40), moment $\overline{E[s_\ell^k]} \stackrel{\text{def}}{=} \lim_{t\to\infty} \langle E[s_\ell(t)^k]\rangle$ for $k = 1, 2$ and covariance $\overline{E[s_\ell s_\nu]} \stackrel{\text{def}}{=} \lim_{t\to\infty} \langle E[s_\ell(t)s_\nu(t)]\rangle$ are obtained based on the MVE of Eq. (13) by using the following equations:

$$\begin{aligned} \overline{E[s_\ell]} &= E^*[s_\ell] \\ &= \sum_{j=0}^{N} \eta_{\ell;j} E^*[\phi_j(\boldsymbol{s})], \end{aligned} \quad (41)$$

$$\begin{aligned} \overline{E[s_\ell s_\nu]} &= E^*[s_\ell s_\nu] \\ &= \sum_{j=0}^{N} \eta_{\ell\nu;j} E^*[\phi_j(\boldsymbol{s})]. \end{aligned} \quad (42)$$

Here, $E^*[\phi_j(\boldsymbol{s})]$ is an element of $\boldsymbol{x}^*$. The variance and correlation coefficient are obtained in the same manner as in Eqs. (22) and (23).

3.4 Power Spectra for Continuous-time Systems

The power spectrum of $s_\ell(t)$ in Eq. (8) is derived in this section, assuming that Assumption 3 holds.

3.4.1 A Periodogram Based on a Numerical Solution

Let $s_\ell(0)$, $s_\ell(\Delta t)$, $s_\ell(2\Delta t)$, $\cdots$ be a sequence obtained by solving Eq. (8) numerically and sampling solution $s_\ell(t)$ at periodic intervals Δt. By rewriting $s_\ell(n\Delta t)$ as $s_\ell(n)$ and applying $s_\ell(n)$ to Eqs. (30) and (31), we can derive periodogram $S_{\ell\ell}(k)$, as an estimation of the power spectrum of $s_\ell(t)$ based on a numerical solution.

3.4.2 A Periodogram Based on an MVE

Correlation function $r_{\ell\nu}(\tau)$ of $s_\ell(t)$ and $s_\nu(t)$ is defined by the following equation:

$$r_{\ell\nu}(\tau) \stackrel{\text{def}}{=} \lim_{t\to\infty} \langle E[s_\ell(t)s_\nu(t+\tau)]\rangle. \tag{43}$$

The right-hand side of the above equation is derived from Eq. (47) in Sect. 3.4.3. Let $r_{\ell\nu}(0)$, $r_{\ell\nu}(\Delta t)$, $r_{\ell\nu}(2\Delta t)$, $\cdots$ be a sequence obtained by sampling $r_{\ell\nu}(\tau)$ at periodic intervals Δt. Let us rewrite $r_{\ell\nu}(m\Delta t)$ as $r_{\ell\nu}(m)$ and apply $r_{\ell\nu}(m)$ to Eq. (33). Then we obtain periodgram $\hat{S}_{\ell\ell}(k)$ based on an MVE ††.

3.4.3 The MVE of the Correlation Function

The right-hand side of Eq. (43) is derived in this section. First, let us expand $E[s_\ell(t)s_\nu(t+\tau)]$ by using Eq. (39) as follows:

$$\begin{aligned} E[s_\ell(t)s_\nu(t+\tau)] &= E[s_\ell(t)\sum_{j=0}^{N}\eta_{\nu;j}\phi_j(\boldsymbol{s}(t+\tau))] \\ &= \eta_{\nu;0}\phi_0 E[s_\ell(t)] \\ &\quad +\sum_{j=1}^{N}\eta_{\nu;j}E[s_\ell(t)\phi_j(\boldsymbol{s}(t+\tau))] \\ &= \eta_{\nu;0}\phi_0 E[s_\ell(t)]+\sum_{j=1}^{N}\eta_{\nu;j}\hat{\boldsymbol{x}}_\ell(\tau;t), \end{aligned} \tag{44}$$

where

$$\hat{\boldsymbol{x}}_\ell(\tau;t) \stackrel{\text{def}}{=} \begin{bmatrix} E[s_\ell(t)\phi_1(\boldsymbol{s}(t+\tau))] \\ \vdots \\ E[s_\ell(t)\phi_N(\boldsymbol{s}(t+\tau))] \end{bmatrix}.$$

Next, let us derive $\hat{\boldsymbol{x}}_\ell(\tau;t)$. From the linearity of $E[\cdot]$, the following equation holds:

$$\mathrm{d}E[s_\ell(t)\phi_i(\boldsymbol{s}(t+\tau))]/\mathrm{d}\tau = E[s_\ell(t)\mathrm{d}\phi_i(\boldsymbol{s}(t+\tau))/\mathrm{d}\tau].$$

Thus, we obtain the following equation for $\tau \geq 0$ by replacing $g_\ell(\boldsymbol{s}(t+\tau))$ with $s_\ell(t)$ in Eq. (A· 13) in Appendix C:

$$\mathrm{d}\hat{\boldsymbol{x}}_\ell(\tau;t)/\mathrm{d}\tau = \boldsymbol{A}\hat{\boldsymbol{x}}_\ell(\tau;t) + E[s_\ell(t)]\boldsymbol{B}.$$

By solving the above equation, we obtain the following equation [8]:

$$\begin{aligned} \hat{\boldsymbol{x}}_\ell(\tau;t) &= \boldsymbol{M}\mathrm{diag}[e^{\lambda_i\tau}]\boldsymbol{M}^{-1}\hat{\boldsymbol{x}}_\ell(0;t) \\ &\quad +\boldsymbol{M}\mathrm{diag}[\lambda_i^{-1}(e^{\lambda_i\tau}-1)]\boldsymbol{M}^{-1}E[s_\ell(t)]\boldsymbol{B}. \end{aligned} \tag{45}$$

Let us consider the time average of the above equation at the limit as t tends to ∞ †. Then we obtain the following equation:

$$\begin{aligned} \bar{\hat{\boldsymbol{x}}}_\ell(\tau) &= \boldsymbol{M}\mathrm{diag}[e^{\lambda_i\tau}]\boldsymbol{M}^{-1}\bar{\hat{\boldsymbol{x}}}_\ell(0) \\ &\quad +\boldsymbol{M}\mathrm{diag}[\lambda_i^{-1}(e^{\lambda_i\tau}-1)]\boldsymbol{M}^{-1}E^*[s_\ell]\boldsymbol{B}, \end{aligned} \tag{46}$$

where $\bar{\hat{\boldsymbol{x}}}_\ell(\tau) \stackrel{\text{def}}{=} \lim_{t\to\infty}\langle\hat{\boldsymbol{x}}_\ell(\tau;t)\rangle$. By replacing $\lim_{t\to\infty}\langle E[s_\ell(t)s_\nu(t+\tau)]\rangle$ with $r_{\ell\nu}(\tau)$, $\lim_{t\to\infty}\langle\hat{\boldsymbol{x}}_\ell(\tau;t)\rangle$ with $\bar{\hat{\boldsymbol{x}}}_\ell(\tau)$, and $\lim_{t\to\infty}\langle E[s_\ell(t)]\rangle$ with $E^*[s_\ell]$ in the time average of Eq. (44) for $t \to \infty$, we obtain correlation function $r_{\ell\nu}(\tau)$ as follows:

$$r_{\ell\nu}(\tau) = \eta_{\nu;0}\phi_0 E^*[s_\ell] + \sum_{j=1}^{N}\eta_{\nu;j}\bar{\hat{\boldsymbol{x}}}_\ell(\tau). \tag{47}$$

Initial value $\bar{\hat{\boldsymbol{x}}}_\ell(0)$ of the above equation is derived in Appendix D.2.

4. Performance Evaluation

To evaluate the accuracy of the MVEs for continuous-time systems described in Eq. (13), the statistical properties, such as the mean, standard deviation, and periodogram, of each variable in the following Lorenz equations [11] were investigated:

$$\begin{aligned} \dot{s}_1 &= \alpha_1(-s_1+s_2), \\ \dot{s}_2 &= s_1(\alpha_2-s_3)-s_2, \\ \dot{s}_3 &= s_1s_2-\alpha_3s_3, \end{aligned} \tag{48}$$

where $(\alpha_1,\alpha_2,\alpha_3) = (10,28,8/3)$. The attractor is shown in Fig. 2. Figures 3 through 6 show the statistical properties of Lorenz equations derived based on MVEs (see Sects. 3.3.2 and 3.4.2) for various values of N_ℓ, where N_ℓ for $\ell = 1,2,3$ denotes the degree of the Fourier series (see Appendix A), $(\check{s}_1,\check{s}_2,\check{s}_3) = (-25,-25,0)$, $(T_1,T_2,T_3) = (50,50,50)$, and $W = 512$. To evaluate the accuracy of the statistical properites derived based on the MVE, those based on the numerical solution of Eq. (48) (see Sects. 3.3.1 and 3.4.1) are also shown in these figures (indicated "Num." on the abscissa axis).

From these figures, the mean, standard deviation, and periodogram based on the MVE approach to those

††Although we can obtain the power spectrum by deriving the Fourier transform of Eq. (43) analytically, sequence $r_{\ell\nu}(m)$ obtained by sampling $r_{\ell\nu}(\tau)$ is used here because of the restriction in computing time.

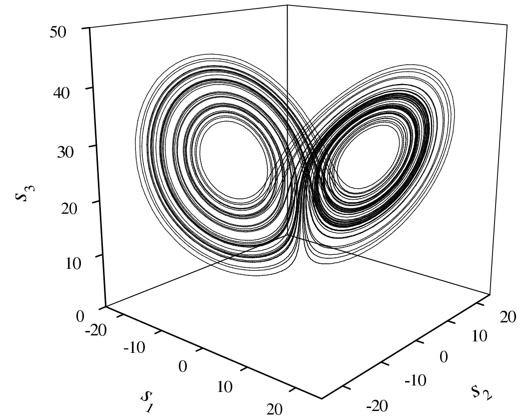

Fig. 2 Lorenz attractor.

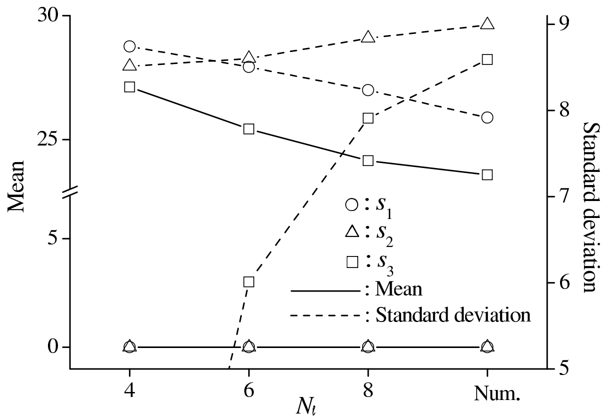

Fig. 3 Mean and standard deviation of Lorenz equations.

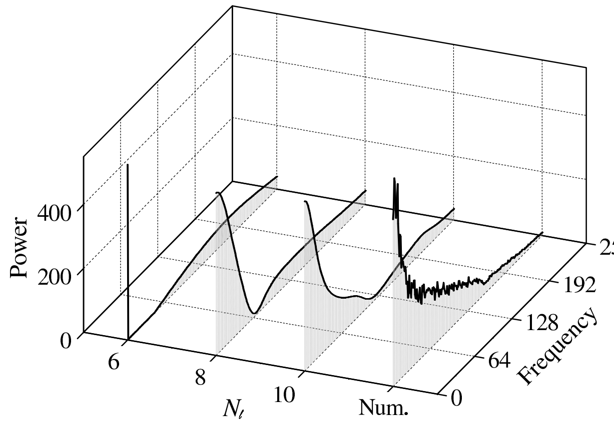

Fig. 4 Periodograms of Lorenz equations (s_1).

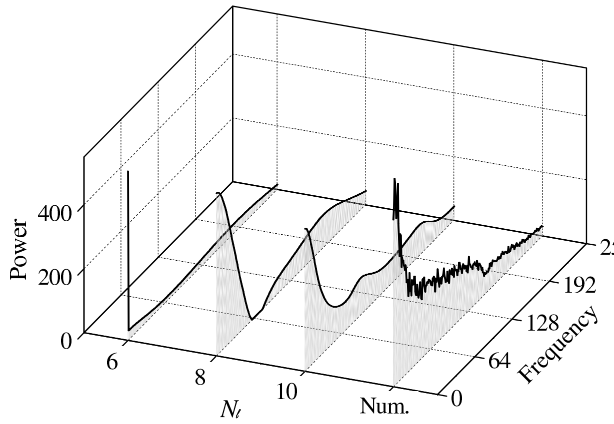

Fig. 5 Periodograms of Lorenz equations (s_2).

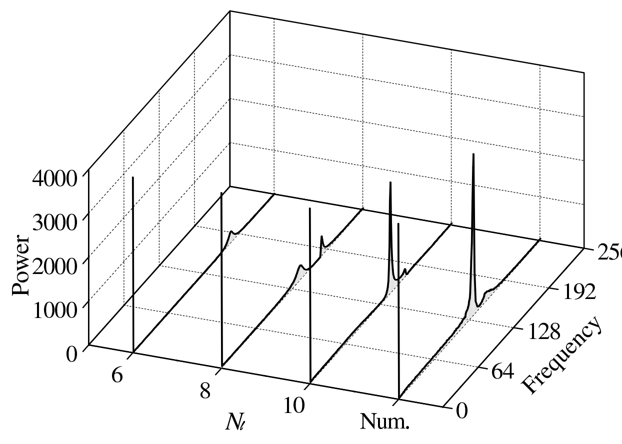

Fig. 6 Periodograms of Lorenz equations (s_3).

based on the numerical solutions as N_ℓ increases. If Assumption 1 holds, that is, the Fourier series converges as N_ℓ increases, the statistical properties converge to the true values as N_ℓ increases. We can also see the difference between the stochastic motions in s_1 and s_2 and the deterministic motion in s_3 from the periodograms. Thus, an MVE can contain statistical properties, such as the mean, standard deviation, and periodogram, of a non-linear equation.

Although we can derive the statistical properties of a non-linear equation based on an MVE, this does not mean that the MVE always approximates the non-linear equation itself. To show that MVEs can be an approximation of non-linear equations, the statistical properties of a combination of the following logistic equations [11] were evaluated by using a combination of their MVEs:

$$\begin{aligned} s_{\mathrm{I}\ell}(n+1) &= f_{\mathrm{I}\ell}(s_{\mathrm{I}\ell}(n)) \\ &\overset{\mathrm{def}}{=} \alpha_\ell s_{\mathrm{I}\ell}(n)(1 - s_{\mathrm{I}\ell}(n)) \quad \text{for} \quad \ell = 1, 2, \end{aligned}$$

where $\alpha_1 = 3.8$ and $\alpha_2 = 3.9$. Consider the following discrete-time non-linear system:

$$\boldsymbol{s}(n+1) = \boldsymbol{f}(\boldsymbol{s}(n)), \tag{49}$$

which is constructed by connecting the above two logistic equations as shown in Eq. (14) and Fig. 1, where

$$\boldsymbol{s}_{\mathrm{II}}(n) \overset{\mathrm{def}}{=} {}^{\mathrm{t}}(s_{\mathrm{II}1}(n), s_{\mathrm{II}2}(n)),$$

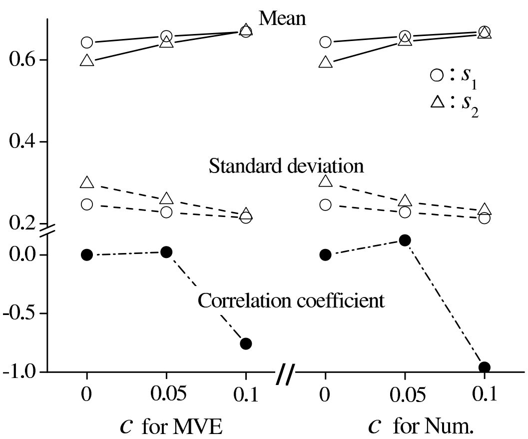

Fig. 7 Moments and correlation coefficient of a combination of logistic equations.

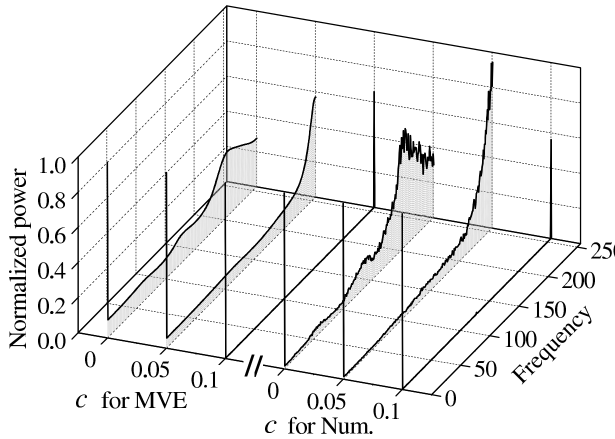

Fig. 8 Power spectrum of a combination of logistic equations (s_1).

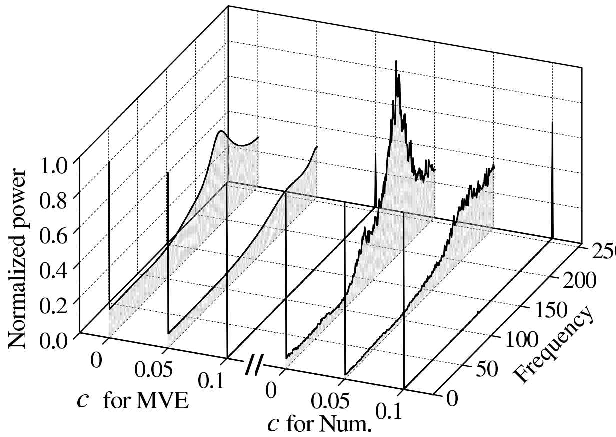

Fig. 9 Power spectrum of a combination of logistic equations (s_2).

$$\begin{aligned}\boldsymbol{s}_{\mathrm{II}}(n+1) &= \boldsymbol{f}_{\mathrm{II}}(\boldsymbol{s}_{\mathrm{II}}(n)) \\ &= \boldsymbol{C}\boldsymbol{s}_{\mathrm{II}}(n),\end{aligned}$$

$$\boldsymbol{C} \stackrel{\mathrm{def}}{=} \begin{bmatrix} 1-c & c \\ c & 1-c \end{bmatrix}.$$

That is, $s_1(n)$ and $s_2(n)$ are independent when $c = 0$, and $s_1(n)$ and $s_2(n)$ affect each other when $c \neq 0$.

Based on Sects. 3.1.2 and 3.2.2, the statistical properties, such as the means, standard deviations, correlation coefficients, and periodograms, of $s_1(n)$ and $s_2(n)$ in Eq. (49) were evaluated using MVEs, which are constructed as shown in Sect. 2.3 from the MVEs of $f_{\mathrm{I}1}$, $f_{\mathrm{I}2}$, and $\boldsymbol{f}_{\mathrm{II}}$. The statistical properties are shown in Figs. 7 through 9 (labeled "MVE" on the abscissa axis), where $W = 512$, $T_\ell = 1$, $\check{s}_\ell = 0$, and $N_\ell = 16$ for $\ell = 1, 2$. The periodograms for $c = 0$ were multiplied by 3, those for $c = 0.05$ were multiplied by 2, and those for $c = 0.1$ were divided by 40 in order to arrange them in the same figures. To evaluate the accuracy of the MVE approach, the statistical properties derived based on the numerical solution of Eq. (49) were also evaluated according to Sects. 3.1.1 and 3.2.1, and they are shown in each figure (labeled "Num." on the abscissa axis).

As shown in the figures, the change in c was reflected both in the statistical properties obtained based on MVEs and in those obtained based on numerical solutions. That is, the means and standard deviations became equal, the correlation coefficients changed from 0 to -1, and the periodograms became line spectra, when the value of c changed from 0 to 0.1. These results show that the statistical properties of a combination of non-linear equations are expressed by using a combination of MVEs of these non-linear equations. Therefore, we can conclude that MVEs are an approximation of non-linear equations in statistical measurements.

5. Conclusion

Moment vector equations (MVEs) can be used to approximate and analyze non-linear equations. We can not only analyze the statistical properties, such as the mean, variance, covariance, and power spectrum, of non-linear equations based on MVEs but also express a combination of non-linear equations by using a combination of MVEs of these equations. Evaluation of the statistical properties of Lorenz equations and those of a combination of logistic equations showed that we can analyze the statistical properties of these equations based on MVEs and that we can use MVEs as an approximation of multi-dimensional non-linear discrete- and continuous-time equations in statistical measurements. Because MVEs can be used to approximate non-linear systems and MVEs are linear, it is expect that we can easily perform stability analysis and control various non-linear systems. I will report on these items in the near future.

Appendix A: The Basis of MVEs

The Fourier series expansion of function $g(\boldsymbol{s})$ with respect to $\boldsymbol{s} \stackrel{\text{def}}{=} {}^{\text{t}}(s_1, \cdots, s_L) \in \boldsymbol{S}$ is defined by [12]

$$g(\boldsymbol{s}) = \sum_{k \in \boldsymbol{Z}} h(\boldsymbol{k}) K(\boldsymbol{s}, \boldsymbol{k}), \tag{A$\cdot$1}$$

$$h(\boldsymbol{k}) \stackrel{\text{def}}{=} \int_{\boldsymbol{S}} g(\boldsymbol{s}) K(\boldsymbol{s}, \boldsymbol{k}) \mathrm{d}\boldsymbol{s}, \tag{A$\cdot$2}$$

where $\boldsymbol{k} \stackrel{\text{def}}{=} {}^{\text{t}}(k_1, \cdots, k_L)$, $\boldsymbol{Z} \stackrel{\text{def}}{=} \{\boldsymbol{k} | 0 \le k_\ell \le N_\ell \text{ for } 1 \le \ell \le L\}$, $h(\boldsymbol{k})$s are Fourier coefficients, and $\{K(\boldsymbol{s}, \boldsymbol{k})\}$ is an orthogonal basis defined by the product of one dimensional orthogonal basis $K_\ell(s_\ell, k_\ell)$ as follows:

$$K(\boldsymbol{s}, \boldsymbol{k}) \stackrel{\text{def}}{=} \prod_{\ell=1}^{L} K_\ell(s_\ell, k_\ell). \tag{A$\cdot$3}$$

Let $\phi_i(\boldsymbol{s})$ be the basis of the MVE defined by

$$\phi_i(\boldsymbol{s}) \stackrel{\text{def}}{=} K(\boldsymbol{s}, \boldsymbol{k}), \tag{A$\cdot$4}$$

where the relationship between i and $\boldsymbol{k} \in \boldsymbol{Z}$ is obtained by the following equation:

$$i = \sum_{\ell=1}^{L} k_\ell \prod_{\nu=\ell+1}^{L} N_\nu. \tag{A$\cdot$5}$$

Dimension N of Matrix $\boldsymbol{A}$ is obtained as follows:

$$N = \prod_{\ell=1}^{L} (N_\ell + 1) - 1.$$

Let $\mathcal{L}(s, k)$ be an orthonormal basis for $s \in [\check{s}, \check{s} + T]$ defined by [12]:

$$\mathcal{L}(s, k) = \sqrt{\frac{2k+1}{T}} P(2\frac{s - \check{s}}{T} - 1, k), \tag{A$\cdot$6}$$

where $P(x, k)$ is the Legendre polynomial for $x \in [-1, 1]$ defined by

$$\begin{aligned} P(x,0) &= 1, \\ P(x,1) &= x, \\ P(x,2) &= (3x^2 - 1)/2, \\ P(x,3) &= (5x^3 - 3x)/2, \\ &\vdots \quad . \end{aligned}$$

In Sect. 4, $K_\ell(s_\ell, k_\ell)$ is set to $\mathcal{L}(s_\ell, k_\ell)$ for $\forall \ell$.

Appendix B: The Average of the Moment Vector

Under Assumptions 2 and 3, although Eqs. (7) and (13) have a unique equilibrium point and do not diverge, the moment vector often oscillates. To derive moments from the equilibrium point even when the moment vector oscillates and to eliminate the effect of the initial value of the moment vector, the relationship between the time average of the moment vector and the equilibrium point is derived in this section.

B.1 The Average for Discrete-time Systems

Let $\boldsymbol{x}^*$ be the equilibrium point of $\boldsymbol{x}(n)$ in Eq. (7). When Assumption 2 holds, $\forall \lambda_i \ne 1$, and $(\boldsymbol{I} - \boldsymbol{A})^{-1}$ exists. Thus, we obtain $\boldsymbol{x}^*$ as follows [8]:

$$\boldsymbol{x}^* = (\boldsymbol{I} - \boldsymbol{A})^{-1} \boldsymbol{B}.$$

The solution of $\boldsymbol{x}(n)$ is expressed by [8]

$$\boldsymbol{x}(n) = \boldsymbol{M} \Lambda^n \boldsymbol{M}^{-1} (\boldsymbol{x}(0) - \boldsymbol{x}^*) + \boldsymbol{x}^*. \tag{A$\cdot$7}$$

From the above equation and Assumption 2, time average $\bar{\boldsymbol{x}}$ is obtained as follows:

$$\begin{aligned} \bar{\boldsymbol{x}} &\stackrel{\text{def}}{=} \lim_{n \to \infty} \langle \boldsymbol{x}(n) \rangle \\ &= \boldsymbol{M} (\lim_{n \to \infty} \langle \Lambda^n \rangle) \boldsymbol{M}^{-1} (\boldsymbol{x}(0) - \boldsymbol{x}^*) + \boldsymbol{x}^* \\ &= \boldsymbol{x}^*. \end{aligned} \tag{A$\cdot$8}$$

B.2 The Average for Continuous-time Systems

Let $\boldsymbol{x}^*$ be the equilibrium point of $\boldsymbol{x}(t)$ in Eq. (13). When Assumption 3 holds, $\boldsymbol{A}^{-1}$ exists, and $\boldsymbol{x}^*$ is expressed by the following equation [8]:

$$\boldsymbol{x}^* = -\boldsymbol{A}^{-1} \boldsymbol{B}. \tag{A$\cdot$9}$$

The solution to Eq. (13) is as follows [8]:

$$\begin{aligned} \boldsymbol{x}(t) = &\ \boldsymbol{M} \mathrm{diag}[e^{\lambda_i t}] \boldsymbol{M}^{-1} \boldsymbol{x}_0 \\ &+ \boldsymbol{M} \mathrm{diag}[\lambda_i^{-1} (e^{\lambda_i t} - 1)] \boldsymbol{M}^{-1} \boldsymbol{B}. \end{aligned} \tag{A$\cdot$10}$$

By using Eqs. (A$\cdot$9) and (A$\cdot$10), we obtain time average $\bar{\boldsymbol{x}}$ as follows:

$$\begin{aligned} \bar{\boldsymbol{x}} &\stackrel{\text{def}}{=} \lim_{t \to \infty} \langle \boldsymbol{x}(t) \rangle \\ &= \lim_{t, \tau_{\max} \to \infty} \frac{1}{\tau_{\max}} (\boldsymbol{M} \mathrm{diag}[\int e^{\lambda_i (t+\tau)} \mathrm{d}\tau] \boldsymbol{M}^{-1} \boldsymbol{x}_0 \\ &\quad + \boldsymbol{M} \mathrm{diag}[\int \frac{1}{\lambda_i} e^{\lambda_i (t+\tau)} \mathrm{d}\tau - \int \frac{1}{\lambda_i} \mathrm{d}\tau] \boldsymbol{M}^{-1} \boldsymbol{B}) \\ &= -\boldsymbol{M} \mathrm{diag}[\lambda_i^{-1}] \boldsymbol{M}^{-1} \boldsymbol{B} \\ &= -\boldsymbol{A}^{-1} \boldsymbol{B} \\ &= \boldsymbol{x}^*. \end{aligned} \tag{A$\cdot$11}$$

Appendix C: The MVE of the Correlation Function

Consider function $g_\ell(\boldsymbol{s})$ and basis $\{h_i(\boldsymbol{s})\}$ with respect to vector variable $\boldsymbol{s}$. For convenience, let us use the following abbreviations both for continuous- and discrete-time systems:

$$\begin{aligned} \boldsymbol{s} &\stackrel{\text{def}}{=} \boldsymbol{s}(n) \ \text{ or } \ \boldsymbol{s}(t+\tau), \\ h_i &\stackrel{\text{def}}{=} h_i(\boldsymbol{s}(n)) \ \text{ or } \ h_i(\boldsymbol{s}(t+\tau)), \\ h_i' &\stackrel{\text{def}}{=} h_i(\boldsymbol{s}(n+1)) \ \text{ or } \ \mathrm{d}h_i(\boldsymbol{s}(t+\tau))/\mathrm{d}\tau, \\ g_\ell &\stackrel{\text{def}}{=} g_\ell(\boldsymbol{s}(n)) \ \text{ or } \ g_\ell(\boldsymbol{s}(t+\tau)). \end{aligned}$$

In this section, correlation function $E[g_\ell h_i']$ is derived. When Assumption 1 holds, we obtain the following equation both for continuous- and discrete-time systems:

$$E[h_i'|\boldsymbol{s}] = \sum_{j=0}^{N} a_{ij} h_j + \varepsilon_i(\boldsymbol{s}). \tag{A· 12}$$

Note that h_0 is a constant. Assuming that $E[\varepsilon_i(\boldsymbol{s})] = 0$ in Eq. (A· 12), we can expand $E[g_\ell h_i']$ as follows:

$$\begin{aligned} E[g_\ell h_i'] &= \int\int g_\ell h_i' p(h_i', \boldsymbol{s}) \mathrm{d}h_i' \mathrm{d}\boldsymbol{s} \\ &= \int g_\ell \int h_i' p(h_i'|\boldsymbol{s}) \mathrm{d}h_i' p(\boldsymbol{s}) \mathrm{d}\boldsymbol{s} \\ &= \int g_\ell E[h_i'|\boldsymbol{s}] p(\boldsymbol{s}) \mathrm{d}\boldsymbol{s} \\ &= \int g_\ell (\sum_{j=0}^{N} a_{ij} h_j) p(\boldsymbol{s}) \mathrm{d}\boldsymbol{s} \\ &= \sum_{j=1}^{N} a_{ij} E[g_\ell h_j] + a_{i0} h_0 E[g_\ell]. \end{aligned}$$

Therefore, using coefficient matrix $\boldsymbol{A}$, the MVE of correlation function $E[g_\ell h_i']$ is expressed as follows:

$$\begin{bmatrix} E[g_\ell h_1'] \\ \vdots \\ E[g_\ell h_N'] \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} E[g_\ell h_1] \\ \vdots \\ E[g_\ell h_N] \end{bmatrix} + E[g_\ell] \begin{bmatrix} a_{10} h_0 \\ \vdots \\ a_{N0} h_0 \end{bmatrix}. \tag{A· 13}$$

Appendix D: The Initial Value of the Correlation Function

Initial values of Eqs. (35) and (47) are derived in this section.

D.1 Discrete-time Systems

We obtain the following equation by expanding $E[s_\ell(n+1)\phi_i(\boldsymbol{s}(n+1))]$ in a series with respect to $E[\phi_j(\boldsymbol{s}(n))]$ as follows:

$$\begin{aligned} E[s_\ell(n+1)\phi_i(\boldsymbol{s}(n+1))] &= E[f_\ell(\boldsymbol{s}(n))\phi_i(\boldsymbol{f}(\boldsymbol{s}(n)))] \\ &= \sum_{j=0}^{N} \xi_{\ell;ij} E[\phi_j(\boldsymbol{s}(n))]. \end{aligned}$$

By using the above equation, Eq. (29), and $\tilde{\boldsymbol{x}}(n)$ defined in Sect. 2.1, $\hat{\boldsymbol{x}}_{\ell\nu}(0; n+1)$ can be expressed by

$$\hat{\boldsymbol{x}}_{\ell\nu}(0; n+1) = \hat{\boldsymbol{\Xi}}_{\ell\nu} \tilde{\boldsymbol{x}}(n), \tag{A· 14}$$

where $\hat{\boldsymbol{\Xi}}_{\ell\nu}$ is defined by

$$\hat{\boldsymbol{\Xi}}_{\ell\nu} \stackrel{\text{def}}{=} \begin{bmatrix} \zeta_{\ell\nu;0} & \zeta_{\ell\nu;1} & \cdots & \zeta_{\ell\nu;N} \\ \xi_{\ell;10} & \xi_{\ell;11} & \cdots & \xi_{\ell;1N} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{\ell;N0} & \xi_{\ell;N1} & \cdots & \xi_{\ell;NN} \end{bmatrix}.$$

Here, $\zeta_{\ell\nu;j}$ is a coefficient used in Eq. (29). Let $\tilde{\boldsymbol{x}}^*$ be the equilibrium point of $\tilde{\boldsymbol{x}}(n)$. Because $\lim_{n\to\infty} \langle \tilde{\boldsymbol{x}}(n) \rangle = \tilde{\boldsymbol{x}}^*$ holds from Appendix B.1, we obtain the initial values of Eq. (35), $\bar{\hat{\boldsymbol{x}}}_{\ell\nu}(0)$, by the following equation:

$$\bar{\hat{\boldsymbol{x}}}_{\ell\nu}(0) = \hat{\boldsymbol{\Xi}}_{\ell\nu} \tilde{\boldsymbol{x}}^*. \tag{A· 15}$$

D.2 Continuous-time Systems

By expanding $E[s_\ell(t)\phi_i(\boldsymbol{s}(t))]$ in a Fourier series with respect to $E[\phi_j(\boldsymbol{s}(t))]$, we obtain

$$E[s_\ell(t)\phi_i(\boldsymbol{s}(t))] = \sum_{j=0}^{N} \xi_{\ell;ij} E[\phi_j(\boldsymbol{s}(t))].$$

Thus, $\hat{\boldsymbol{x}}_\ell(0; t)$ is obtained by

$$\hat{\boldsymbol{x}}_\ell(0; t) = \boldsymbol{\Xi}_\ell \tilde{\boldsymbol{x}}(t),$$

where $\tilde{\boldsymbol{x}}(t) \stackrel{\text{def}}{=} {}^{\mathrm{t}}(E[\phi_0(\boldsymbol{s}(t))], \cdots, E[\phi_N(\boldsymbol{s}(t))])$ and

$$\boldsymbol{\Xi}_\ell \stackrel{\text{def}}{=} \begin{bmatrix} \xi_{\ell;10} & \xi_{\ell;11} & \cdots & \xi_{\ell;1N} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{\ell;N0} & \xi_{\ell;N1} & \cdots & \xi_{\ell;NN} \end{bmatrix}. \tag{A· 16}$$

Let $\tilde{\boldsymbol{x}}^* \stackrel{\text{def}}{=} {}^{\mathrm{t}}(\phi_0, {}^{\mathrm{t}}\boldsymbol{x}^*)$ and $\bar{\tilde{\boldsymbol{x}}}(t) \stackrel{\text{def}}{=} \lim_{t\to\infty} \langle \tilde{\boldsymbol{x}}(t) \rangle$. From Appendix B.2, $\bar{\tilde{\boldsymbol{x}}}(t) = \tilde{\boldsymbol{x}}^*$. Therefore, we can obtain the initial values of Eq. (47), $\bar{\hat{\boldsymbol{x}}}_\ell(0)$, by the following equation:

$$\bar{\hat{\boldsymbol{x}}}_\ell(0) = \boldsymbol{\Xi}_\ell \tilde{\boldsymbol{x}}^*.$$

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